A Review on Harmonic Wavelets and Their Fractional Extension

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Abstract. *In this paper a review on harmonic wavelets and their fractional generalization, within the local fractional calculus, will be discussed. The main properties of harmonic wavelets and fractional harmonic wavelets will be given, by taking into account of their characteristic features in the Fourier domain. It will be shown that the local fractional derivatives of fractional wavelets have a very simple expression thus opening new frontiers in the solution of fractional differential problems.*

Keywords

Harmonic wavelets, local fractional derivative, wavelet series.

1. Introduction

Harmonic wavelets are some kind of complex wavelets [1–9] which are analitically defined, infinitely differentiable, and band-limited in the Fourier domain. Although the slow decay in the space domain, their sharp localization in frequency, is a good property especially for the analysis of wave evolution problems (see e.g. [1–3,10,13,15,16,25,32,33]. In the search for numerical approximation of differential problems, the main idea is to approximate the unknown solution by some wavelet series and then by computing the integrals (or derivatives) of the basic wavelet functions, to convert the starting differential problem into an algebraic system for the wavelet coefficients (see e.g. [26–30]).

Wavelets are some special functions (see e.g. [5, 9, 24]) which depend on two parameters, the scale parameter (also called refinement, compression, or dilation parameter) and a the localization (translation) parameter. These functions fulfill the fundamental axioms of multiresolution analysis so that by a suitable choice of the scale and translation parameter one is able to easily and quickly approximate (almost) all functions (even tabular) with decay to infinity.

Therefore wavelets seems to be the more expedient tool for studying differential problems which are localized (in time or in frequency).

There exists a very large literature devoted to wavelet solution of partial differential and integral equations (see e.g. the pioneristic works [10, 13, 25, 35]) integral equations (see e.g. [11, 23, 34] and more general integro-differential equations and operators (see e.g. [26–30]).

By using the derivatives (or integrals) of the wavelet basis the PDE equation can be transformed into an infinite dimensional system of ordinary differential equations. By fixing the scale of approximation, the projection correspond to the choice of a finite set of wavelet spaces, thus obtaining the numerical (wavelet) approximation.

By using the orthogonality of the wavelet basis and the computation of the inner product of the basis functions with their derivatives or integrals (operational matrix, also called connection coefficients), we can convert the differential problem into an algebraic system and thus we can easily derive the wavelet approximate solution. The approximation depends on the fixed scale (of approximation) and on the number of dilated and translated instances of the wavelets. However, due to their localization property just a few instances are able to capture the main feature of the signal, and for this reason it is enough to compute a few number of wavelet coefficients to quickly get a quite good approximation of the solution.

In recent years there has been a fast rising interest for the fractional differential problems. Indeed the idea of fractional order derivative is deeply rooted in the history of mathematics, since already Cauchy was wondering about the possible generalization of ordinary differential operators to fractional order differential operators. The main advantage of fractional order derivative is to have an additional parameter (the order of derivative) to be use in the analysis of differential problems. On the other hand the main drawback for the fractional differential operators is that this derivative is not univocally defined (see e.g. [19–22] and references therein). We will not go deeply into this subject, since we will focus only on a special fractional operator, the so-called local fractional derivative, as defined by Yang [12, 31, 36, 37].

The local fractional derivative when applied to the most popular functions give a natural generalization of known results and fulfills the basica axioms of the fractional calculus.

In the following after reviewing on the classical Harmonic wavelet, the fractional harmonic wavelets will be defined. Moreover their local fractional derivatives will be explicitly computed. It will be shown that these fractional derivatives, are some kind of generalization already obtained for the so called Shannon wavelets [17, 18] and the sinc-derivative [19, 20, 22]

The paper is organized as follows: in section 2 some preliminary definitions about harmonic (complex wavelets) together with their fractional counterparts are given. The harmonic wavelet reconstruction of functions is described in section 3. In the same section, the harmonic wavelet representation of the fractional harmonic functions will be also given. Section 4 shows some characteristic features of harmonic wavelets. In section 5 the basic definitions and properties of local fractional derivatives are given and the local fractional derivatives of the fractional harmonic wavelets will be explicitly computed.

2. Harmonic (Newland) Wavelets

Harmonic wavelets also known as Newland wavelets [1, 3, 5, 7, 8] are complex orthonormal wavelets that are characterized by the sharply bounded frequency and slow decay in the space of variable. Like any other wavelet they depend both on the scale parameter n which define the degree of refinement, compression, or dilation and on a second parameter k which is related to the space localization. As we will see, harmonic wavelets fulfill the fundamental axioms of multiresolution analysis (see e.g. [24]), but they also enjoy some more special features especially in the function approximation.

2.1. Harmonic scaling function

The harmonic scaling function is defined as

$$\boxed{\varphi(x) \stackrel{\text{def}}{=} \frac{e^{2\pi i x} - 1}{2\pi i x}} \tag{1}$$

that is

$$\varphi(x) = \frac{\sin(2\pi x)}{2\pi x} + i \left[\frac{1 - \cos(2\pi x)}{2\pi x} \right]$$

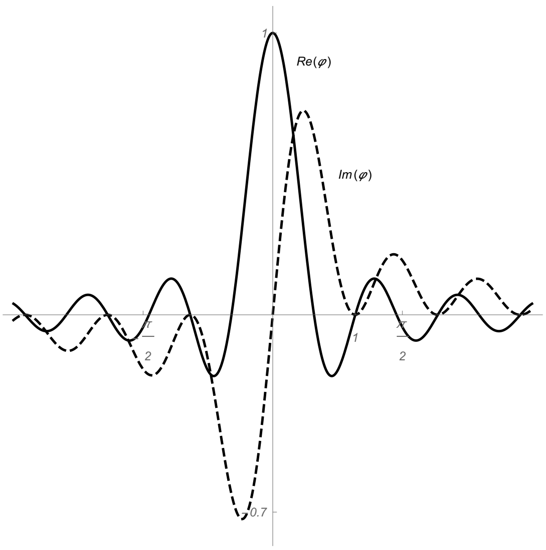

Fig. 1: Plot of the scaling function in the complex plane ($0 \leq x \leq 4$).

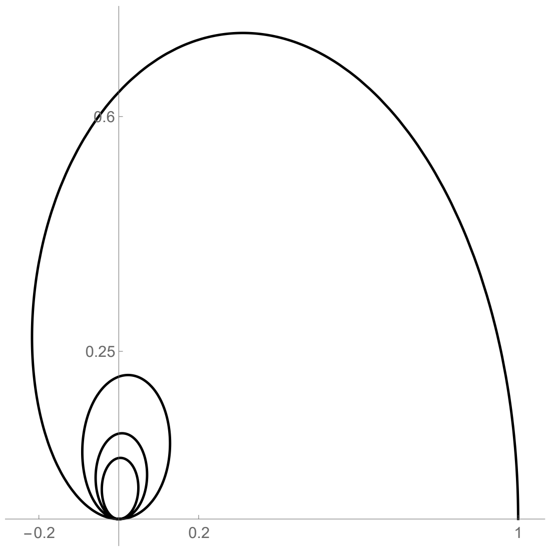

Fig. 2: Plot of the scaling function in the complex plane ($0 \leq x \leq 4$).

there follow the real and imaginary part of the scaling function

$$\begin{aligned} \varphi_r(x) &\stackrel{\text{def}}{=} \Re[\varphi(x)] = \frac{\sin(2\pi x)}{2\pi x}, \\ \varphi_i(x) &\stackrel{\text{def}}{=} \Im[\varphi(x)] = \frac{1-\cos(2\pi x)}{2\pi x}. \end{aligned} \tag{2}$$

Plots of real $\varphi_r(x)$ and imaginary part , $\varphi_i(x)\}$ of the scaling function in the real plane are shown in Fig. 1. The parametric plot $\{\varphi_r(x), \varphi_i(x)\}$ of the complex scaling function $\varphi(x)$ is shown in Fig. 2.

It can be easily seen that

$$\lim_{x\to\infty} \varphi_r(x) = \lim_{x\to\infty} \varphi_i(x) = 0$$

and

$$\lim_{x\to 0} \varphi_r(x) = 1, \quad \lim_{x\to 0} \varphi_i(x) = 0$$

Moreover, since

$$e^{\pi i n} = \begin{cases} 1, & n = 2k, \quad k \in \mathbb{Z} \\ -1, & n = 2k+1, \quad k \in \mathbb{Z} \end{cases} \tag{3}$$

it is, in particular,

$$\varphi(n) = 0, \qquad n \in \mathbb{Z}. \tag{4}$$

The complex conjugate of the function $\varphi(x)$ is the function

$$\overline{\varphi}(x) = \frac{1-e^{-2\pi i x}}{2\pi i x}. \tag{5}$$

2.2. Fractional prolungation of the scaling function

The scaling function (1) is the power series, with complex coefficients,

$$\varphi(x) = \frac{e^{2\pi i x}-1}{2\pi i x} = \sum_{k=0}^{\infty} \frac{(2\pi i)^k}{(k+1)!} x^k \tag{6}$$

Let us slightly modify the harmonic scaling function by using the Mittag-Leffler function, instead of the exponential. So that we have

$$\varphi_\alpha(x) \stackrel{\text{def}}{=} \frac{E_\alpha(2\alpha\pi i x)-1}{2\pi i x}, \quad (0 \leq \alpha \leq 1) \tag{7}$$

being

$$E_\alpha(x) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(\alpha k+1)}. \tag{8}$$

the Mittag-Leffler function.

When $\alpha = 1$, namely we have

$$\varphi_1(x) \to \varphi(x)$$

while for $\alpha = 0$, it is

$$\varphi_0(x) \to \delta(x)$$

where $\delta(x)$ is the Dirac delta

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

By a direct computation we have the *fractional scaling function*

$$\boxed{\varphi_\alpha(x) \stackrel{\text{def}}{=} \frac{E^{2\pi\alpha i x} - 1}{2\pi\alpha i x} = \sum_{k=0}^{\infty} \frac{(2\pi i)^k}{\alpha\Gamma(k+\alpha+1)} x^k, \quad (0 \leq \alpha \leq 1)} \tag{9}$$

2.3. Scaling function in Fourier domain

The Fourier transform of the scaling function (1) is defined as

$$\widehat{\varphi}(\omega) = \widehat{\varphi(x)} \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(x) e^{-i\omega x} \mathrm{d}x.$$

So that, in the frequency domain, i.e. with respect to the variable ω the Fourier transform is a function with a compact support (i.e. with a bounded frequency)

$$\boxed{\widehat{\varphi}(\omega) = \frac{1}{2\pi} \chi(2\pi + \omega)} \tag{10}$$

$\chi(\omega)$ being the characteristic function defined as

$$\chi(\omega) \stackrel{\text{def}}{=} \begin{cases} 1, & 2\pi \leq \omega \leq 4\pi, \\ 0, & \text{elsewhere.} \end{cases} \tag{11}$$

The scaling function in Fourier domain is box-function thus being defined in a sharp domain with slow decay in frequency.

The Fourier transform of the fractional scaling function (9) can be also computed so that we have at the first approximation

$$\widehat{\varphi}(\omega) = \frac{2\pi}{\alpha\Gamma(1+\alpha)} \delta(\omega) \tag{12}$$

2.4. Harmonic wavelet function

Theorem 1. *The harmonic (Newland) wavelet function is defined as [3, 4, 7, 8]*

$$\boxed{\psi(x) \stackrel{\text{def}}{=} \frac{e^{4\pi i x} - e^{2\pi i x}}{2\pi i x} = e^{2\pi i x} \varphi(x)} \tag{13}$$

and its Fourier transform is

$$\boxed{\widehat{\psi}(\omega) = \frac{1}{2\pi} \chi(\omega)} \tag{14}$$

Proof. Starting from $\varphi(x)$ we have to define a filter and to derive the corresponding wavelet function (see e.g. [7]). From (10) we have

$$\begin{aligned} \widehat{\varphi}(\omega) &= \frac{1}{2\pi} \chi(2\pi + \omega) \chi(2\pi + \frac{\omega}{2}) \\ &= \chi(2\pi + \omega) \widehat{\varphi}(\frac{\omega}{2}) \end{aligned} \tag{15}$$

so that,

$$\widehat{\varphi}(\omega) = H\left(\frac{\omega}{2}\right) \widehat{\varphi}\left(\frac{\omega}{2}\right)$$

with

$$H\left(\frac{\omega}{2}\right) = \chi(2\pi + \omega).$$

In order to have a multiresolution analysis [3, 5, 7, 24] the wavelet function must be defined as (see e.g. [24])

$$\widehat{\psi}(\omega) = \overline{H\left(\frac{\omega}{2} \pm 2\pi\right)} \widehat{\varphi}\left(\frac{\omega}{2}\right)$$

where the bar stands for complex conjugation.

With the filter $H\left(\frac{\omega}{2} - 2\pi\right) = \chi(\omega)$ we have

$$\begin{aligned} \widehat{\psi}(\omega) &= \overline{H\left(\frac{\omega}{2} - 2\pi\right)} \widehat{\varphi}\left(\frac{\omega}{2}\right) \\ &= \chi(\omega) \frac{1}{2\pi} \chi\left(2\pi + \frac{\omega}{2}\right) \\ &= \frac{1}{2\pi} \chi(\omega) \end{aligned}$$

while with $H\left(\frac{\omega}{2} + 2\pi\right)$ we obtain

$$\widehat{\psi}(\omega) = \frac{1}{2\pi} \chi(4\pi + \omega) \chi(2\pi + \frac{\omega}{2}) = 0 \qquad \forall\omega$$

from where there follows (14).

By the inverse Fourier transform of (14) we get

$$\int_{-\infty}^{\infty}\frac{1}{2\pi}\chi(\omega)e^{i\omega x}\mathrm{d}\omega=\frac{1}{2\pi}\int_{2\pi}^{4\pi}e^{i\omega x}\mathrm{d}\omega,$$

we get the harmonic wavelet (13).

□

The real and imaginary parts of (13) are:

$$\begin{cases}\Re\left(\psi(x)\right)=\dfrac{\left(e^{4\pi ix}-e^{2\pi ix}+e^{-2\pi ix}-e^{-4\pi ix}\right)}{4\pi ix}\\ \qquad\qquad=\dfrac{\sin 4\pi x}{2\pi x}-\dfrac{\sin 2\pi x}{2\pi x},\\ \Im\left(\psi(x)\right)=\dfrac{\left(-e^{4\pi ix}+e^{2\pi ix}+e^{-2\pi ix}-e^{-4\pi ix}\right)}{4\pi x}\\ \qquad\qquad=-\dfrac{\cos 4\pi x}{2\pi x}+\dfrac{\cos 2\pi x}{2\pi x}.\end{cases}$$

In particular, according to (3), (4), (13) it is

$$|\psi(x)|=|\varphi(x)|=\left|\frac{\sin\pi x}{\pi x}\right|,\quad \psi(n)=0,\quad n\in\mathbb{Z}.$$

The complex conjugate of the function $\psi(x)$ is the function

$$\overline{\psi}(x)=\frac{e^{-2\pi ix}-e^{-4\pi ix}}{2\pi ix}. \tag{16}$$

2.5. Fractional prolungation of the harmonic wavelet

From Eqs. (13), (8) we can define the fractional prolungation of the harmonic wavelet as

$$\psi_\alpha(x)\stackrel{\text{def}}{=}e^{2\pi\alpha ix}\varphi_\alpha(x) \tag{17}$$

and its Fourier transform is

$$\widehat{\psi}_\alpha(\omega)=\frac{2\pi}{\alpha\Gamma(1+\alpha)}\delta(2\alpha^2\pi-\omega) \tag{18}$$

2.6. Dilated and translated instances

In order to have a family of (harmonic) wavelet functions we have to define the dilated (compressed) and translated instances of the fundamental functions (1), (13), so that there will be a family of functions depending on the scaling parameter n and on the translation paremater k. From Eqs. (1), (13) there immediately follows (see e.g. [1,3,7,8]),

Theorem 2. *The dilated and translated instances of the harmonic scaling and wavelet function are*

$$\boxed{\begin{cases}\varphi_k^n(x)\stackrel{\text{def}}{=}2^{n/2}\dfrac{e^{2\pi i\,(2^n x-k)}-1}{2\pi i(2^n x-k)}\\ \psi_k^n(x)\stackrel{\text{def}}{=}2^{n/2}\dfrac{e^{4\pi i(2^n x-k)}-e^{2\pi i(2^n x-k)}}{2\pi i(2^n x-k)}\end{cases}} \tag{19}$$

with $n,k\in\mathbb{Z}$.

For each function of the wavelet family (19), it is $|\psi_k^n(x)|=\left|\dfrac{\sin\pi\,(2^n x-k)}{\pi\,(2^n x-k)}\right|$ so that $\lim\limits_{n,k,x\to\infty}|\psi_k^n(x)|=0$.

Let us now compute the Fourier transform of the parameter depending instances (19), by using the properties of the Fourier transform. It is known that if $\widehat{f}(\omega)$ is the Fourier transform of $f(x)$ then

$$\widehat{f(ax\pm b)}=\frac{1}{a}e^{\pm i\omega b/a}\widehat{f}(\omega/a)\ , \tag{20}$$

so that we can easily obtain the dilated and translated instances of the Fourier transform of (19), (see e.g. [3]):

$$\boxed{\begin{cases}\widehat{\varphi}_k^n(\omega)=\dfrac{2^{-n/2}}{2\pi}e^{-i\omega k/2^n}\chi(2\pi+\omega/2^n)\\ \widehat{\psi}_k^n(\omega)=\dfrac{2^{-n/2}}{2\pi}e^{-i\omega k/2^n}\chi(\omega/2^n)\end{cases}} \tag{21}$$

3. Multiscale harmonic wavelet reconstruction of functions

In this section we give the inner product space structure to the family of harmonic wavelets

(19) and the harmonic wavelet reconstruction of functions.

3.1. Hilbert space structure

Let $f(x)$, $g(x)$ be given two complex functions, the inner (or scalar or dot) product, of these functions is

$$\langle f,g\rangle \overset{\text{def}}{=} \int_{-\infty}^{\infty} f(x)\overline{g(x)}\mathrm{d}x \overset{Pars.}{=} 2\pi \int_{-\infty}^{\infty} \widehat{f}(\omega)\overline{\widehat{g}(\omega)}\mathrm{d}\omega = 2\pi \left\langle \widehat{f},\widehat{g}\right\rangle, \tag{22}$$

where we have used the Parseval identity for the equivalent inner product in the Fourier domain.

With respect to the family of the fundamental functions (19), it can be shown that

Theorem 3. *Harmonic wavelets are orthonormal functions, such that*

$$\langle \psi_k^n(x), \psi_h^m(x)\rangle = \delta^{nm}\delta_{hk}, \tag{23}$$

where δ^{nm} (δ_{hk}) is the Kronecker symbol.

Proof: It is (for an alternative proof see also [7])

$$\begin{aligned}&\langle \psi_k^n(x), \psi_h^m(x)\rangle \\ &= 2\pi\int_{-\infty}^{\infty} \frac{2^{-n/2}}{2\pi} e^{-i\omega k/2^n}\chi(\omega/2^n)\frac{2^{-m/2}}{2\pi} \\ &\times e^{i\omega h/2^m}\chi(\omega/2^m)\mathrm{d}\omega \\ &= \frac{2^{-(n+m)/2}}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega k/2^n}\chi(\omega/2^n) \\ &\times e^{i\omega h/2^m}\chi(\omega/2^m)\mathrm{d}\omega\end{aligned}$$

which is zero for $n \neq m$. For $n = m$ it is

$$\begin{aligned}&\langle \psi_k^n(x), \psi_h^n(x)\rangle \\ &= \frac{2^{-n}}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(h-k)/2^n}\chi(\omega/2^n)\mathrm{d}\omega.\end{aligned}$$

Moreover, according to (11), by the change of variable $\xi = \omega/2^n$

$$\langle \psi_k^n(x), \psi_h^n(x)\rangle = \frac{1}{2\pi}\int_{2\pi}^{4\pi} e^{-i(h-k)\xi}\mathrm{d}\xi.$$

For $h = k$ (and $n = m$), trivially one has: $\langle \psi_k^n(x), \psi_k^n(x)\rangle = 1$, while for $h \neq k$, it is

$$\int_{2\pi}^{4\pi} e^{-i(h-k)\xi}\mathrm{d}\xi = \frac{i}{(h-k)}\left(e^{-4i\pi(h-k)} - e^{-2i\pi(h-k)}\right).$$

and since, according to (3),

$$e^{\pm 4i\pi(h-k)} = e^{\pm 2i\pi(h-k)} = 1, \quad (h-k \in \mathbb{Z}), \tag{24}$$

the proof easily follows. □

Analogously it can be easily shown that

$$\left\{\begin{aligned}&\langle \varphi_k^n(x), \varphi_h^m(x)\rangle = \delta^{nm}\delta_{kh}, \\ &\qquad\langle \overline{\varphi}_k^n(x), \overline{\varphi}_h^m(x)\rangle = \delta^{nm}\delta_{kh}, \\ &\qquad\langle \varphi_k^n(x), \overline{\varphi}_h^m(x)\rangle = 0, \\ &\left\langle \overline{\psi}_k^n(x), \overline{\psi}_h^m(x)\right\rangle = \delta^{nm}\delta_{kh}, \\ &\qquad\left\langle \psi_k^n(x), \overline{\psi}_h^m(x)\right\rangle = 0, \\ &\left\langle \varphi_k^n(x), \overline{\psi}_h^m(x)\right\rangle = 0, \\ &\qquad\langle \overline{\varphi}_k^n(x), \psi_h^m(x)\rangle = 0.\end{aligned}\right. \tag{25}$$

Moreover, the fundamental functions (1), (13) fulfills the basic (even-odd) properties of scaling and wavelet, that is

$$\begin{aligned}&\Re[\varphi(x)] = \Re[\varphi(-x)], \qquad \Im[\varphi(x)] = -\Im[\varphi(-x)] \\ &\Re[\psi(x)] = -\Re[\psi(-x)], \quad \Im[\psi(x)] = \Im[\psi(-x)]\end{aligned}$$

and the following

Theorem 4. *The harmonic scaling function and the harmonic wavelets fulfill the conditions*

$$\int_{-\infty}^{\infty} \varphi(x)dx = 1, \qquad \int_{-\infty}^{\infty} \psi_k^n(x)dx = 0.$$

Proof: According to (10)-(22) one has

$$\int_{-\infty}^{\infty} \varphi(x)\mathrm{d}x$$
$$= \langle 1, \varphi(x)\rangle = 2\pi \left\langle \widehat{1}, \widehat{\varphi}(\omega)\right\rangle$$
$$= 2\pi \int_{-\infty}^{\infty} \delta(\omega)\frac{1}{2\pi}\chi(2\pi+\omega)\mathrm{d}\omega$$
$$= \int_{0}^{2\pi} \delta(\omega)\mathrm{d}\omega = 1,$$

where $\delta(\omega)$ is the Dirac delta function.

Analogously, taking into account (21)-(22),

$$\int_{-\infty}^{\infty} \psi_k^n(x)\mathrm{d}x$$
$$= \langle 1, \psi_k^n(x)\rangle = 2\pi \left\langle \widehat{1}, \widehat{\psi}_k^n(\omega)\right\rangle$$
$$= 2\pi \int_{-\infty}^{\infty} \delta(\omega)\frac{2^{-n/2}}{2\pi}e^{-i\omega k/2^n}\chi(\omega/2^n)\mathrm{d}\omega$$
$$= \int_{2^{n+1}\pi}^{2^{n+2}\pi} \delta(\omega)e^{-i\omega k/2^n}\mathrm{d}\omega = 0.$$

□

3.2. Wavelet reconstruction

Let $f(x) \in \mathcal{B}$, where $\mathcal{B}$ is the space of complex functions, such that for any value of the parameters n, k, the following integrals, which define the wavelet coefficients, exist and have finite values

$$\begin{cases} \alpha_k = \langle f(x), \varphi_k^0(x)\rangle = \int_{-\infty}^{\infty} f(x)\overline{\varphi}_k^0(x)\mathrm{d}x \\ \alpha_k^* = \langle f(x), \overline{\varphi}_k^0(x)\rangle = \int_{-\infty}^{\infty} f(x)\varphi_k^0(x)\mathrm{d}x \\ \beta_k^n = \langle f(x), \psi_k^n(x)\rangle = \int_{-\infty}^{\infty} f(x)\overline{\psi}_k^n(x)\mathrm{d}x \\ \beta^{*}{}_k^n = \langle f(x), \overline{\psi}_k^n(x)\rangle = \int_{-\infty}^{\infty} f(x)\psi_k^n(x)\mathrm{d}x. \end{cases} \tag{26}$$

According to (21),(22), these coefficients can be equivalently computed in the Fourier domain, thus being

$$\begin{cases} \alpha_k &= 2\pi\langle \widehat{f(x)}, \widehat{\varphi_k^0(x)}\rangle \\ &= \int_{-\infty}^{\infty} \widehat{f}(\omega)\overline{\widehat{\varphi}_k^0(\omega)}\mathrm{d}\omega = \int_0^{2\pi} \widehat{f}(\omega)e^{i\omega k}\mathrm{d}\omega \\ \alpha_k^* &= 2\pi\langle \widehat{f(x)}, \widehat{\overline{\varphi}_k^0(x)}\rangle \\ &= \ldots = \int_0^{2\pi} \widehat{f}(\omega)e^{-i\omega k}\mathrm{d}\omega \\ \beta_k^n &= 2\pi\langle \widehat{f(x)}, \widehat{\psi_k^n(x)}\rangle \\ &= \ldots = 2^{-n/2}\int_{2^{n+1}\pi}^{2^{n+2}\pi} \widehat{f}(\omega)e^{i\omega k/2^n}\mathrm{d}\omega \\ \beta^{*}{}_k^n &= \langle \widehat{f(x)}, \widehat{\overline{\psi}_k^n(x)}\rangle \\ &= \ldots = 2^{-n/2}\int_{2^{n+1}\pi}^{2^{n+2}\pi} \widehat{f}(\omega)e^{-i\omega k/2^n}\mathrm{d}\omega, \end{cases} \tag{27}$$

where the hat stands for the Fourier transform. It can be easily seen (see e.g. [14]) that

$$\widehat{\overline{f(x)}} = \overline{\widehat{f}(-\omega)}.$$

3.3. Harmonic wavelet series

Let $f(x) \in \mathcal{B}$ be a complex funtion with finite wavelet coefficients (26), (27). By taking into account the orthonormality of the basis functions (23), (25) the function $f(x)$ can be expressed as a wavelet (convergent) series (see e.g. [7]). In fact, if we put

$$f(x) = \left[\sum_{k=-\infty}^{\infty} \alpha_k\varphi_k^0(x) + \sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} \beta_k^n\psi_k^n(x)\right] + \left[\sum_{k=-\infty}^{\infty} \alpha_k^*\overline{\varphi}_k^0(x) + \sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} \beta^{*}{}_k^n\overline{\psi}_k^n(x)\right] \tag{28}$$

the wavelet coefficients can be easily computed by using the orthogonality of the basis and its conjugate.

In [7] (see also [24]) it was shown that, under suitable and quite general hypotheses on the function $f(x)$, the wavelet series (28) converges to $f(x)$.

The conjugate of the reconstruction (28) it is

$$\begin{aligned}\overline{f}(x) &= \left[\sum_{k=-\infty}^{\infty} \overline{\alpha}_k \overline{\varphi}_k^0(x) + \sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} \overline{\beta}_k^n \overline{\psi}_k^n(x)\right] \\ &+ \left[\sum_{k=-\infty}^{\infty} \overline{\alpha}_k^* \varphi_k^0(x) + \sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} \overline{\beta^*}_k^n \psi_k^n(x)\right] \\ &= \left[\sum_{k=-\infty}^{\infty} \overline{\alpha}_k^* \varphi_k^0(x) + \sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} \overline{\beta^*}_k^n \psi_k^n(x)\right] \\ &+ \left[\sum_{k=-\infty}^{\infty} \overline{\alpha}_k \overline{\varphi}_k^0(x) + \sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} \overline{\beta}_k^n \overline{\psi}_k^n(x)\right]\end{aligned}$$

The wavelet approximation is obtained by fixing an upper limit in the series expansion (28), so that with $N < \infty,\ M < \infty$ we have

$$\begin{aligned}f(x) &\cong \left[\sum_{k=0}^{M} \alpha_k \varphi_k^0(x) + \sum_{n=0}^{N}\sum_{k=-M}^{M} \beta_k^n \psi_k^n(x)\right] \\ &+ \left[\sum_{k=0}^{M} \alpha_k^* \overline{\varphi}_k^0(x) + \sum_{n=0}^{N}\sum_{k=-M}^{M} {\beta^*}_k^n \overline{\psi}_k^n(x)\right].\end{aligned} \tag{29}$$

Since wavelets are localized, they can capture with few terms the main features of functions defined in a short range interval.

1) Examples of Harmonic wavelet reconstruction

Let us give a couple of examples to show the powerful approximation obtained by the harmonic wavelets.

Let us first consider the reconstruction of the Gaussian function:

$$f(x) = e^{-x^2/\sigma}\ .$$

The truncated wavelet series with $N = 0\ , M = 0$ is

$$f(x) \cong \alpha_0 \varphi_0^0(x) + \alpha_0^* \overline{\varphi}_0^0(x) + \beta_0^0 \psi_0^0 + {\beta^*}_0^0 \overline{\psi}_0^0,$$

so that if we compute the wavelet coefficients $\alpha_0\ , \alpha_0^*\ , \beta_0^0\ , {\beta^*}_0^0$ by using the Eqs. (26) (or (27)) we get

$$\begin{aligned}\alpha_0 &= \alpha_0^* = \tfrac{1}{2}\ \mathrm{erf}\,(\pi\sqrt{\sigma}), \\ \beta_0^0 &= {\beta^*}_0^0 = \tfrac{1}{2}\left[\ \mathrm{erf}\,(2\pi\sqrt{\sigma}) -\ \mathrm{erf}\,(\pi\sqrt{\sigma})\right]\end{aligned}$$

being the error function defined as

$$\mathrm{erf}\,(x) \stackrel{\text{def}}{=} \frac{2}{\sqrt{\pi}} \int_0^x e^{-u} du$$

There follows the zero order approximation of the Gaussian

$$\begin{aligned}f(x) &\cong \frac{1}{2}\ \mathrm{erf}\,(\pi\sqrt{\sigma})\left[\varphi_0^0(x) + \overline{\varphi}_0^0(x)\right] \\ &+ \frac{1}{2}\left[\ \mathrm{erf}\,(2\pi\sqrt{\sigma}) -\ \mathrm{erf}\,(\pi\sqrt{\sigma})\right] \\ &\times \left[\psi_0^0(x) + \overline{\psi}_0^0(x)\right],\end{aligned}$$

and since

$$\varphi_0^0(x) + \overline{\varphi}_0^0(x) = \frac{\sin 2\pi x}{x}$$

and

$$\psi 0_0(x) + \overline{\psi}_0^0(x) = \frac{\sin 4\pi x - \sin 2\pi x}{\pi x}$$

we have

$$\begin{aligned}e^{-x^2/\sigma} &\cong \frac{1}{2}\ \mathrm{erf}\,(\pi\sqrt{\sigma}) \frac{\sin 2\pi x}{x} \\ &+ \frac{1}{2}\left[\ \mathrm{erf}\,(2\pi\sqrt{\sigma}) -\ \mathrm{erf}\,(\pi\sqrt{\sigma})\right] \\ &\times \frac{\sin 4\pi x - \sin 2\pi x}{\pi x}\end{aligned}$$

For instance, the second scale approximation $N = 2,\ M = 0$ for the Gaussian function $e^{-(16x)^2}$ is (see Fig. 3)

$$\begin{aligned}e^{-(16x)^2} &\cong \frac{\sin 2\pi x}{2\pi x}\Bigg[\left(2\ \mathrm{erf}\,\frac{\pi}{16} -\ \mathrm{erf}\,\frac{\pi}{8}\right) \\ &- 2\cos 2\pi x\left(\ \mathrm{erf}\,\frac{\pi}{16} -\ \mathrm{erf}\,\frac{\pi}{8}\right) \\ &- 2\cos 6\pi x\left(\ \mathrm{erf}\,\frac{\pi}{8} -\ \mathrm{erf}\,\frac{\pi}{4}\right) \\ &- (\cos 10\pi x + \cos 14\pi x) \\ &\times \left(\ \mathrm{erf}\,\frac{\pi}{4} -\ \mathrm{erf}\,\frac{\pi}{2}\right)\Bigg]\end{aligned}$$

As expected, by increasing the scaling parameter N we will get a better approximation.

2) Computation of the wavelet coefficients in the Fourier domain

According to (27) the wavelet coefficient are obtained by Fourier transform.

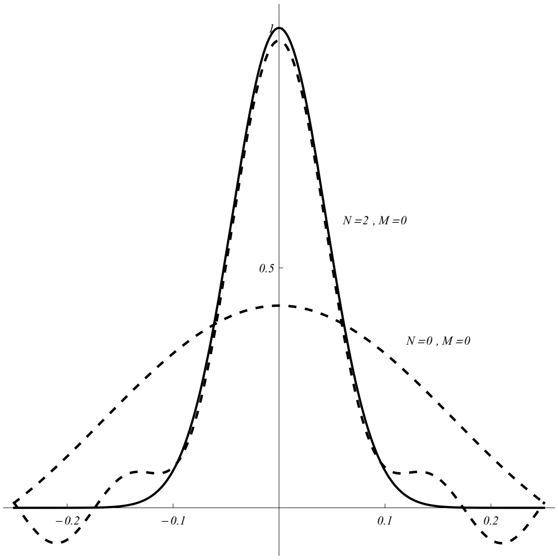

Fig. 3: Harmonic wavelet approximation of the function $f(x) = e^{-(16x)^2}$ and the 0-scale $N = 0, M = 0$ and 2-scale $N = 2, M = 0$ approximation.

If we apply the Fourier transform to (29), we get

$$\widehat{f}(\omega) \cong \left[\sum_{k=0}^{M} \alpha_k \widehat{\varphi}_k^0(\omega) + \sum_{n=-N}^{N} \sum_{k=-M}^{M} \beta_k^n \widehat{\psi}_k^n(\omega)\right] + \left[\sum_{k=0}^{M} \alpha_k^* \widehat{\overline{\varphi}}_k^0(\omega) + \sum_{n=-N}^{N} \sum_{k=-M}^{M} \beta^{*n}_k \widehat{\overline{\psi}}_k^n(\omega)\right]$$

and, according to (21),

$$\widehat{f}(\omega) \cong \left[\frac{1}{2\pi} \sum_{k=0}^{M} \alpha_k e^{-i\omega k} \chi(2\pi+\omega) + \sum_{n=0}^{N} \frac{2^{-n/2}}{2\pi} \times \sum_{k=-M}^{M} \beta_k^n e^{-i\omega k/2^n} \chi(2\pi + \omega/2^n)\right] + \left[\frac{1}{2\pi} \sum_{k=0}^{M} \alpha_k^* e^{i\omega k} \chi(2\pi+\omega) + \sum_{n=0}^{N} \frac{2^{-n/2}}{2\pi} \times \sum_{k=-M}^{M} \beta^{*n}_k e^{i\omega k/2^n} \chi(2\pi + \omega/2^n)\right]$$

i.e.

$$\widehat{f}(\omega) \cong \left[\frac{1}{2\pi} \chi(2\pi+\omega) \sum_{k=0}^{M} \alpha_k e^{-i\omega k} + \sum_{n=0}^{N} \frac{2^{-n/2}}{2\pi} \times \chi(2\pi + \omega/2^n) \sum_{k=-M}^{M} \beta_k^n e^{-i\omega k/2^n}\right] + \left[\frac{1}{2\pi} \chi(2\pi+\omega) \sum_{k=0}^{M} \alpha_k^* e^{i\omega k} + \sum_{n=0}^{N} \frac{2^{-n/2}}{2\pi} \times \chi(2\pi + \omega/2^n) \sum_{k=-M}^{M} \beta^{*n}_k e^{i\omega k/2^n}\right]$$

and for a real function

$$\widehat{f}(\omega) \cong \frac{1}{2\pi} \chi(2\pi+\omega) \sum_{k=0}^{M} \alpha_k \left(e^{-i\omega k} + e^{i\omega k}\right) + \sum_{n=0}^{N} \frac{2^{-n/2}}{2\pi} \chi(2\pi + \omega/2^n) \times \sum_{k=-M}^{M} \beta_k^n \left(e^{-i\omega k/2^n} + e^{i\omega k/2^n}\right)$$

that is,

$$\widehat{f}(\omega) \cong \frac{1}{2\pi} \chi(\pi+\omega) \sum_{k=0}^{M} \alpha_k \cos(\omega k) + \sum_{n=0}^{N} \frac{2^{-n/2}}{\pi} \chi(2\pi + \omega/2^n) \times \sum_{k=-M}^{M} \beta_k^n \cos(\omega k/2^n)$$

So that the wavelet coefficient can be obtained by the fast Fourier transform. In [7] it was given a simple algorithm for the computation of these coefficients through the fast Fourier transform.

3) Harmonic wavelet coefficients of the fractional harmonic scaling and wavelet

The fractional harmonic scaling and wavelet functions (9), (17) in general are not orthogonal as can be checked by a direct computation of their inner product. However, they can be expressed, by the wavelet coefficients with respect to the harmonic wavelet basis. By taking into account the simple form of the Fourier transform of the fractional functions (12), (18)

$$\begin{aligned}\widehat{\varphi}_\alpha(\omega) &= \frac{2\pi}{\alpha\Gamma(1+\alpha)}\delta(\omega),\\ \widehat{\psi}_\alpha(\omega) &= \frac{2\pi}{\alpha\Gamma(1+\alpha)}\delta(2\alpha^2\pi-\omega)\end{aligned} \tag{30}$$

we have for the scaling function $\varphi_\alpha(x)$

$$\left\{\begin{aligned}\alpha_k &= \int_0^{2\pi}\widehat{\varphi}_\alpha(\omega)e^{i\omega k}\mathrm{d}\omega = \frac{2\pi}{\alpha\Gamma(1+\alpha)}\\ \alpha_k^* &= \int_0^{2\pi}\widehat{\varphi}_\alpha(\omega)e^{-i\omega k}\mathrm{d}\omega = \frac{2\pi}{\alpha\Gamma(1+\alpha)}\\ \beta_k^n &= 2^{-n/2}\int_{2^{n+1}\pi}^{2^{n+2}\pi}\widehat{\varphi}_\alpha(\omega)e^{i\omega k/2^n}\mathrm{d}\omega\\ &= \frac{2\pi}{\alpha\Gamma(1+\alpha)}\\ \beta^{*n}_k &= 2^{-n/2}\int_{2^{n+1}\pi}^{2^{n+2}\pi}\widehat{\varphi}_\alpha(\omega)e^{-i\omega k/2^n}\mathrm{d}\omega\\ &= \frac{2\pi}{\alpha\Gamma(1+\alpha)}\ ,\end{aligned}\right. \tag{31}$$

Analogously for the fractional wavelet $\psi_\alpha(x)$

$$\left\{\begin{aligned}\alpha_k &= \int_0^{2\pi}\widehat{f}(\omega)e^{i\omega k}\mathrm{d}\omega = \frac{2\pi}{\alpha\Gamma(1+\alpha)}e^{2\pi i\alpha^2 k}\\ \alpha_k^* &= \int_0^{2\pi}\widehat{f}(\omega)e^{-i\omega k}\mathrm{d}\omega = \frac{2\pi}{\alpha\Gamma(1+\alpha)}e^{2\pi i\alpha^2 k}\\ \beta_k^n &= 2^{-n/2}\int_{2^{n+1}\pi}^{2^{n+2}\pi}\widehat{f}(\omega)e^{i\omega k/2^n}\mathrm{d}\omega\\ &= \frac{2\pi}{\alpha\Gamma(1+\alpha)}e^{2\pi i\alpha^2 k}\\ \beta^{*n}_k &= 2^{-n/2}\int_{2^{n+1}\pi}^{2^{n+2}\pi}\widehat{f}(\omega)e^{-i\omega k/2^n}\mathrm{d}\omega\\ &= \frac{2\pi}{\alpha\Gamma(1+\alpha)}e^{2\pi i\alpha^2 k}\ ,\end{aligned}\right. \tag{32}$$

So that according to (28) we get the fractional scaling as a wavelet series

$$\boxed{\varphi_\alpha(x) = \frac{2\pi}{\alpha\Gamma(1+\alpha)}\sum_{k=-\infty}^{\infty}\left[\varphi_k^0(x)+\overline{\varphi}_k^0(x)\right]} \tag{33}$$

and analogously for the fractional harmonic wavelet

$$\boxed{\begin{aligned}\psi_\alpha(x) = \frac{2\pi}{\alpha\Gamma(1+\alpha)}\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} e^{2\pi i\alpha^2 k}\\ \times\left[\psi_k^n(x)+\overline{\psi}_k^n(x)\right].\end{aligned}} \tag{34}$$

By taking into account Eqs.(1), (5), the basic functions on the right hand side can be simplified thus giving

$$\varphi_\alpha(x) = \frac{4\pi}{\alpha\Gamma(1+\alpha)}\sum_{k=-\infty}^{\infty}\frac{\sin 2\pi(x-k)}{2\pi(x-k)} \tag{35}$$

so that the fractional scaling is closely related to the sinc-fractional operator (see e.g. [22]) and for the fractional wavelet, from (13), (16), analogously we get

$$\begin{aligned}\varphi_\alpha(x) = \frac{2\pi}{\alpha\Gamma(1+\alpha)}\sum_{k=-\infty}^{\infty} e^{2\pi i\alpha^2 k}\\ \times\left[\frac{\sin 4\pi(x-k)}{\pi(x-k)}-\frac{\sin 2\pi(x-k)}{\pi(x-k)}\right]\end{aligned} \tag{36}$$

Also the fractional wavelet is closely related to the Shannon wavelet and the sinc-fractional wavelets [22].

4. Some properties of the Harmonic wavelets in Fourier domain

It is clear from (27) that the reconstruction of a function $f(x)$ it is impossible when its Fourier transform $\widehat{f}(\omega)$ is not defined. Moreover, the function (to be reconstructed) must be concentrated around the origin (like a pulse) and should rapidly decay to zero. The reconstruction can be done also for periodic functions, or functions localized in a point different from zero: $x_0 \neq 0$, by using the so-called periodized harmonic wavelets [1, 7, 8]).

Among all functions $f(x)$ some of them are constant under harmonic wavelet map (28). In fact, we have that,

Theorem 5. *For a non trivial function $f(x) \neq 0$ the corresponding wavelet coefficients (27), in general, vanish when either*

$$\widehat{f}(\omega) = 0,\ \forall k \quad \text{or} \quad \widehat{f}(\omega) = Cnst.,\ k \neq 0.$$

In particular, it can be seen that the wavelet coefficients (27) trivially vanish when

$$\begin{cases} f(x) = \sin(2k\pi x), & k \in \mathbb{Z} \\ f(x) = \cos(2k\pi x), & k \in \mathbb{Z} \quad (k \neq 0) \end{cases} \tag{37}$$

Proof: For instance from $(26)_1$, for $\cos(2k\pi x)$ it is

$$\begin{aligned} \alpha_k &= \int_{-\infty}^{\infty} \cos(2k\pi x)\overline{\varphi}_k^0(x)\mathrm{d}x \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \left(e^{-2ih\pi x} + e^{2ih\pi x}\right)\overline{\varphi}_k^0(x)\mathrm{d}x \\ &= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{-2ih\pi x}\overline{\varphi}_k^0(x)\mathrm{d}x \right. \\ &\qquad \left. + \int_{-\infty}^{\infty} e^{2ih\pi x}\overline{\varphi}_k^0(x)\mathrm{d}x\right] \end{aligned}$$

from where by the change of variable $2\pi x = \xi$ and taking into account (20) there follows

$$\alpha_k = \frac{1}{2}\left[\widehat{\overline{\varphi}_k^0(x)} + \widehat{\varphi}_k^0(x)\right]_{x=2\pi h}.$$

According to (21) it is

$$\begin{aligned} \widehat{\varphi}_k^0(2\pi h) &= \frac{1}{2}e^{-i2\pi hk}\chi(2\pi + 2\pi h) \\ &\overset{(3)}{=} \frac{1}{2}\chi(2\pi + 2\pi h) \end{aligned}$$

and, because of (11)

$$\chi(2\pi + 2\pi h) = 1, \quad 0 < h < 1$$

so that

$$\widehat{\varphi}_k^0(2\pi h) = 0, \quad \forall h \neq 0.$$

There follows that $\alpha_h = 0$, as well as the remaining wavelet coefficients of $\cos(2k\pi x)$ (with $k \in \mathbb{Z}$ and $k \neq 0$) are trivially vanishing. Analogously, it can be shown that all wavelet coefficients of $\cos(2k\pi x)$ ($\forall k \in \mathbb{Z}$) are zero.

□

As a consequence, a given function $f(x)$, for which the coefficients (26) are defined, admits the same wavelet coefficients of

$$f(x) + \sum_{h=0}^{\infty}\left[A_h \sin(2h\pi x) + B_h \cos(2h\pi x)\right] - B_0, \tag{38}$$

or (by a simple tranformation) in terms of complex exponentials,

$$f(x) - C_0 + \sum_{h=-\infty}^{\infty} C_h e^{2ih\pi x}, \tag{39}$$

so that the wavelet coefficients of $f(x)$ are defined unless an additional trigonometric series (the coefficients A_h , B_h , C_h being constant) as in (38).

5. Local fractional calculus

In order to get some advantages from the definition of the fractional harmonic wavelets we give in this section the definition of the local fraction derivative, and then we apply this operator to the fractional wavelets (9), (17). By taking into account that wavelets are localized functions, we need to define a suitable local differential operator as the ones proposed by Yang [36–39]:

5.1. Local fractional derivative

Definition 1. *The local fractional derivative of $f(x)$ of order α at $x = x_0$ is the operator*

$$\left.\frac{d^\alpha f}{dx^\alpha}\right|_{x=x_0} = \lim_{x\to x_0}\frac{\Delta^\alpha(f(x) - f(x_0))}{(x^\alpha - x_0^\alpha)}, \quad 0 < \alpha \leq 1 \tag{40}$$

being

$$\Delta^\alpha(f(x) - f(x_0)) \cong \Gamma(1+\alpha)\left[(f(x) - f(x_0))\right]. \tag{41}$$

There follows that

$$\begin{aligned} \left.\frac{d^\alpha x^\alpha}{dx^\alpha}\right|_{x=x_0} &= \lim_{x\to x_0} \frac{\Delta^\alpha(f(x) - f(x_0))}{(x^\alpha - x_0^\alpha)} \\ &\cong \Gamma(1+\alpha) \lim_{x\to x_0} \frac{(x^\alpha - x_0^\alpha)}{(x^\alpha - x_0^\alpha)} \\ &= \Gamma(1+\alpha) \end{aligned}$$

i.e.,

$$d^\alpha x^\alpha = \Gamma(1+\alpha) dx^\alpha$$

For any x in a suitable interval centered in x_0, we can define the local fractional derivative

$$D_x^\alpha f(x) \stackrel{\text{def}}{=} \frac{d^\alpha}{dx^\alpha} f(x), \qquad x \in (x_0 - \delta, x_0 + \delta)$$

5.2. Local fractional integral

Definition 2. *The local fractional integral of $f(x)$ of fractional order α in the interval (a, b) is defined as ([36, 37])*

$$\begin{aligned} {}_aI_b^{(\alpha)} f(x) &= \frac{1}{\Gamma(1+\alpha)} \int_a^b f(u)(du)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta u \longrightarrow 0} \sum_{j=0}^{N-1} f(u_j)(\Delta u_j)^\alpha, \end{aligned} \tag{42}$$

where we have $\Delta u_j = u_{j+1} - u_j$, $\Delta u = \max\{\Delta u_0, \Delta u_1, \Delta u_2, \cdots\}$ and $[u_j, u_{j+1}]$, $u_0 = a$, $u_N = b$, is a partition of the interval $[a, b]$. For any $x \in (a, b)$, we can also define the integral operator ${}_aI_x^{(\alpha)} f(x)$,

5.3. Some properties of the local fractional operators

The local fractional operators, previously defined, have some special features when applied to the most significant functions. By a direct computation it can be easily shown that, starting from the power series [31, 36–39]:

$$E_\alpha(x^\alpha) = \sum_{m=0}^{+\infty} \frac{x^{m\alpha}}{\Gamma(1+m\alpha)}, \qquad 0 < \alpha \le 1, \tag{43}$$

$$\sin_\alpha(x^\alpha) = \sum_{m=0}^{+\infty} (-1)^m \frac{x^{(2m+1)\alpha}}{\Gamma(1+(2m+1)\alpha)}, \quad 0 < \alpha \le 1 \tag{44}$$

$$\cos_\alpha(x^\alpha) = \sum_{m=0}^{+\infty} (-1)^m \frac{x^{2m\alpha}}{\Gamma(1+2m\alpha)}, \quad 0 < \alpha \le 1 \tag{45}$$

and by taking into account that [36, 37]

$$\boxed{\frac{d^\alpha x^{m\alpha}}{dx^\alpha} = \frac{\Gamma(1+m\alpha)}{\Gamma(1+(m-1)\alpha)} x^{(m-1)\alpha}.} \tag{46}$$

we can easily show that

$$\frac{d^\alpha}{dx^\alpha} E_\alpha(x^\alpha) = E_\alpha(x^\alpha). \tag{47}$$

$$\frac{d^\alpha}{dx^\alpha} \sin_\alpha(x^\alpha) = \cos_\alpha(x^\alpha). \tag{48}$$

$$\frac{d^\alpha}{dx^\alpha} \cos_\alpha(x^\alpha) = -\sin_\alpha(x^\alpha). \tag{49}$$

$${}_0I_x^{(\alpha)} \frac{x^{m\alpha}}{\Gamma(1+m\alpha)} = \frac{x^{(m+1)\alpha}}{\Gamma(1+(m+1)\alpha)}. \tag{50}$$

5.4. Local fractional derivative of the fractional Harmonic wavelets

In this section we will give the explicit expression of the local fractional derivative of the harmonic fractional scaling (9) and wavelet (17), namely

$$\varphi_\alpha(x) = \sum_{k=0}^{\infty} \frac{(2\pi i)^k}{\alpha\Gamma(k+\alpha+1)} x^k, \quad (0 \le \alpha \le 1)$$
$$\psi_\alpha(x) = E(2\pi\alpha i x)\varphi_\alpha(x) \tag{51}$$

According to Eqs. (46), (47) it is

$$\frac{d}{dx^\alpha}\varphi_\alpha(x) = \sum_{k=0}^{\infty} \frac{(2\pi i)^k}{\alpha\Gamma(k+\alpha+1)} \frac{d}{dx^\alpha} x^k$$
$$= \sum_{k=1}^{\infty} \frac{(2\pi i)^k}{\alpha\Gamma(k+\alpha+1)} \frac{\Gamma(1+k)}{\Gamma(k)} x^{k-1}, \quad (0 \le \alpha \le 1)$$
$$\frac{d}{dx^\alpha}\psi_\alpha(x) = E_\alpha(2\pi\alpha i x) \times \left(2\pi\alpha i \varphi_\alpha(x) + \frac{d}{dx^\alpha}\varphi_\alpha(x) \right) \tag{52}$$

and symplifying

$$\frac{d}{dx^\alpha}\varphi_\alpha(x) = \sum_{k=1}^{\infty} \frac{k(2\pi i)^k}{\alpha\Gamma(k+\alpha+1)} x^{k-1}, \quad (0 \le \alpha \le 1)$$
$$\frac{d}{dx^\alpha}\psi_\alpha(x) = E_\alpha(2\pi\alpha i x) \left(\sum_{k=0}^{\infty} \frac{(2\pi i)^{k+1}}{\Gamma(k+\alpha+1)} x^k + \sum_{k=1}^{\infty} \frac{k(2\pi i)^k}{\alpha\Gamma(k+\alpha+1)} x^{k-1} \right) \tag{53}$$

The knowledge of the local fractional derivative of the fractional harmonic wavelets can be a fundamental tool in the search for numerical solution of fractional differential equations.

Conclusion

In this paper the main properties of the complex harmonic wavelets are given. Moreover the fractional harmonic wavelets were defined and their local fractional derivatives explicitly computed. These fractional harmonic wavelets are the fundamental functions to build a model for the solution of fractional differential problems.

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