Asymptotic Behavior of Bounded Solutions to a First Order Gradient-like System

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(Received: 26-Dec-2018; accepted: 25-Feb-2019; published: 31-Mar-2019)
DOI: http://dx.doi.org/10.25073/jaec.201931.227

Abstract. *In this paper, we prove the long time behavior of bounded solutions to a first order gradient-like system with low damping and perturbation terms. Our convergence results are obtained under some hypotheses of Kurdyka-Lojasiewicz inequality and the angle and comparability condition.*

Keywords

Asymptotic Behavior, Gradient-Like System, Convergence to Equilibrium.

1. Introduction

The main goal of this paper is to obtain the asymptotic behavior of bounded solutions to the gradient-like system as follows

$$u'(t)+\gamma(t)u(t)+G(u(t)) = f(t),\ t \in [0,\infty), \quad (1)$$

where the unknown $u(t) \in \mathbb{R}^n$, the damping term $\gamma \in L^1(\mathbb{R}^+,\mathbb{R}^+)$, the perturbation term $f \in L^1(\mathbb{R}^+,\mathbb{R}^n)$ and $G \in C(\mathbb{R}^n,\mathbb{R}^n)$ is a tangent vector field on $\mathbb{R}^n$. Roughly speaking, we study in this paper the effect of adding a low damping term $\gamma(t)u(t)$ and a perturbation term $f(t)$ to the equation

$$u'(t) + G(u(t)) = 0,\ t \in [0,\infty), \quad (2)$$

on the long time behavior of the trajectories u.

This type of problem have been studied in many recent papers with different assumptions of G. The typical situation of (2) is the case of gradient system when $G = \nabla F$. This gradient system was studied by many authors such as [1], [6], [14], [15], [18] or [21]. In the classical result, they proved that the bounded solution converges to an equilibrium as t goes to infinity if the function F is real analytic in [18]. More later, R. Chill *et al.* [8] established an general result which guarantees that the convergence result also holds for the gradient-like system (2). This convergence result was proved under the hypotheses of the Lojasiewicz inequality of F and the angle condition of G and ∇F. In [19] and [20], the authors extended the result by Kurdyka-Lojasiewicz inequality. Moreover, the convergence rates was obtained if F satisfies Lojasiewicz inequality and $G, \nabla F$ satisfy angle and comparability condition.

Recently, R. Chill and M. Jendoubi [7] or Huang and Takac [17] considered the equation in the non homogeneous case. They showed that any bounded solution of the gradient system

$$\dot{u} + \nabla F(u) \ = \ f(t), \quad t \geq 0, \quad (3)$$

converges to a critical point of F at infinity under the following condition

$$\sup_{t\in\mathbb{R}^+} t^{1+\mu}\int_0^\infty \|f(s)\|^2 \mathrm{d}s \ < \ \infty, \qquad (4)$$

for some positive constant μ. The forcing term $f(t)$ quickly decays to zero as t goes to infinity in this case. In a previous work [22], we proved the convergence result of (3) under a low L^1-condition of the perturbation term. Moreover, the rate of convergence was even obtained under a Lojasiewicz inequality of the Lyapunov function F. The convergence results have been generalized to some second order systems, such as [2], [4], [5], [12] or [13] are references therein. Moreover, M. Ghisi *et. al.* have estimated the decay rates for solutions of semi linear dissipative equations in [9] and [10].

Motivated by these works, we establish in this paper the convergence results for the first order non homogeneous gradient-like system (1) with the effect of a low damping and forcing terms. More precisely, we consider the equation (1) with γ and f satisfy the following condition

$$\int_0^\infty \left(\int_t^\infty \gamma^2(s)\mathrm{d}s\right)^{1/2} \mathrm{d}t < \infty,$$

and

$$\int_0^\infty \left(\int_t^\infty \|f(s)\|^2\mathrm{d}s\right)^{1/2} \mathrm{d}t < \infty.$$

In addition, we also assume some key hypotheses such as the angle and comparability condition of G and ∇F and the Kurdyka-Lojasiewicz inequality of F as in many other articles. The nice feature of angle and comparability condition of G and ∇F is that G is coincident with the gradient of F with respect to a Riemannian metric g. We refer the reader to the article of Barta *et. al.* [3] for the detail. Under these assumptions, we prove that the bounded solution u to equation (1) converges to a critical point $\varphi \in \omega[u]$ at infinity and $\dot{u} \in L^1(\mathbb{R}^+)$.

The paper is organized as follows. In the next section, we present some assumptions and definitions that we use through the whole of the paper. We also recall the existence of a Riemannian metric g such that $G = \nabla_g F$ in this section. In the last section, we establish the asymptotic behavior of bounded solutions to gradient-like system (1). Our results are divided into three theorems for the convenience of the reader.

2. Preliminaries

In this section, we give the key assumption of angle and comparability condition to obtain the convergence result of gradient-like system. We also recall some definitions about the Lyapunov function, the Kurdyka-Lojasiewicz inequality and the gradient of a function with respect to a Riemannian metric.

We consider a continuous tangent vector field $G \in C(\mathbb{R}^n, \mathbb{R}^n)$ respects to a Lyapunov function $F \in C^1(\mathbb{R}^n, \mathbb{R})$. In this paper, we always assume the angle and comparability condition of ∇F and G, i.e, there exists a positive constant $a > 0$ such that for any $u \in \mathbb{R}^n$ there holds

$$\langle G(u), \nabla F(u)\rangle \geq a\left(\|G(u)\|^2 + \|\nabla F(u)\|^2\right), \qquad (5)$$

where $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the inner product and Euclidean norm in $\mathbb{R}^n$ respectively.

Definition 1. *F is called a Lyapunov function for equation (1) if*

$$\langle G(u), \nabla F(u)\rangle \geq 0, \ \forall u \in \mathbb{R}^n.$$

Moreover, we say that F is a strict Lyapunov function if $\nabla F(u) = 0$ implies $G(u) = 0$.

We remark that F is a strict Lyapunov function if the angle and comparability condition (5) holds.

Definition 2. *We say that the function F satisfies a Kurdyka-Lojasiewicz inequality at η if there exist $\delta > 0$ and a non decreasing function $\Theta \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that*

$$\Theta(0) = 0, \ \Theta^{-1} \in L^1_{loc}(\mathbb{R}^+), \qquad (6)$$

and

$$\Theta\left(|F(u) - F(\eta)|\right) \leq \|\nabla F(u)\|, \ \forall u \in B_\delta(\eta), \qquad (7)$$

where $B_\delta(\eta)$ denotes the ball centered at η and radius δ in $\mathbb{R}^n$.

This definition is related to the Lojasiewicz Theorem in [18] below.

Theorem 1. *If F is real analytic in a neighborhood of η then F satisfies the Kurdyka-Lojasiewicz inequality (7) at η.*

We denote by $\nabla_{g(u)}F(u)$ the gradient of F with respect to a Riemannian metric g on $\mathbb{R}^n$ at u, i.e., for any $v \in \mathbb{R}^n$, we have

$$\langle \nabla F(u), v\rangle = \langle \nabla_{g(u)}F(u), v\rangle_{g(u)}. \tag{8}$$

For simplicity, we write $\nabla_g F(u)$ instead of $\nabla_{g(u)}F(u)$ and $\langle\cdot,\cdot\rangle_g$ instead of $\langle\cdot,\cdot\rangle_{g(u)}$. We denote by $\|\cdot\|_g$ the induced norm. In [3], the authors showed that there exists a Riemannian metric g which is equivalent to the Euclidean metric such that $G = \nabla_g F$. We recall this result in the following theorem.

Theorem 2. *Assume the angle and comparability condition (5) of ∇F and G holds. Then there exists a Riemannian metric g on*

$$V := \{u \in \mathbb{R}^n : \; G(u) \neq 0\}$$

such that $G = \nabla_g F$.

Moreover, there exist positive constants α, β such that

$$\alpha\|v\| \le \|v\|_{g(u)} \le \beta\|v\|, \tag{9}$$

for any $v \in \mathbb{R}^n$, $u \in V$ and $\|\cdot\|$ denotes the Euclidean metric on $\mathbb{R}^n$.

Definition 3. *For any trajectory u belongs to $C(\mathbb{R}^+, \mathbb{R}^n)$, the ω-limit set of u is defined by*

$$\omega[u] = \{\varphi \in \mathbb{R}^n : \; \exists (t_m) \uparrow \infty, \; u(t_m) \to \varphi\}.$$

3. Main results

In this section, we prove the convergence of bounded solutions to equilibrium of the gradient-like system (1). The main idea of our work is based on Theorem 2. Applying Theorem 2, the gradient-like system (1) can be seen as a form of gradient system.

Theorem 3. *Let u be a bounded solution of (1) and f, $\gamma \in L^2(\mathbb{R}^+)$. Assume that the angle and comparability condition (5) of G and ∇F holds. Then $\dot{u} \in L^2(\mathbb{R}^+)$ and $G(u) \in L^2(\mathbb{R}^+)$.*

Proof. Let us consider the energy function defined by

$$\Phi(t) = F(u(t)) + \frac{1}{2}\int_t^\infty \|\gamma(s)u(s) - f(s)\|^2_{g(u(s))}\mathrm{d}s. \tag{10}$$

The function Φ is well defined. Indeed, applying Theorem 2 for G and ∇F under the angle and comparability condition (5), we have $G = \nabla_g F$ and the inequality (9) holds. Combining with $f, \gamma \in L^2(\mathbb{R}^+)$, we then obtain that

$$\begin{aligned}
&\int_t^\infty \|\gamma(s)u(s) - f(s)\|^2_{g(u(s))}\mathrm{d}s \\
&\quad \le \beta^2 \int_t^\infty \|\gamma(s)u(s) - f(s)\|^2 \mathrm{d}s \\
&\quad \le 2\beta^2 \sup_{t\in[0,\infty)} \|u(t)\| \int_t^\infty |\gamma(s)|^2 \mathrm{d}s \\
&\qquad + 2\beta^2 \int_t^\infty \|f(s)\|^2 \mathrm{d}s \\
&\quad \le C\left(\|\gamma\|^2_{L^2(\mathbb{R}^+)} + \|f\|^2_{L^2(\mathbb{R}^+)}\right) \\
&\quad < \infty,
\end{aligned}$$

where $C = 2\beta^2 \max\{1, \sup_{t\in[0,\infty)} \|u(t)\|\}$.

For every $t \in \mathbb{R}^+$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) &= \langle \nabla F(u(t)), \dot{u}\rangle \\
&\qquad - \frac{1}{2}\|\gamma(t)u(t) - f(t)\|^2_g \\
&= \langle \nabla_g F(u(t)), \dot{u}\rangle_g \\
&\qquad - \frac{1}{2}\|\gamma(t)u(t) - f(t)\|^2_g.
\end{aligned}$$

By the angle and comparability (5), equation (1) and Theorem 2, we can estimate the derivative of the energy function Φ as follows

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) &= \langle G(u(t)), \dot{u}\rangle_g - \frac{1}{2}\|\dot{u} + G(u(t))\|^2_g \\
&= -\frac{1}{2}\left(\|\dot{u}\|^2_g + \|G(u(t))\|^2_g\right) \\
&\le -\frac{a}{2}\left(\|\dot{u}\|^2 + \|G(u(t))\|^2\right) \\
&\le 0.
\end{aligned}$$

It deduces that Φ is non increasing and therefore

$$\begin{aligned}\int_0^\infty \|\dot{u}(s)\|^2\mathrm{d}s &\leq -\frac{2}{a}\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\Phi(s)\mathrm{d}s \\ &\leq \frac{2}{a}\left(\Phi(0)-\Phi(\infty)\right) \\ &< \infty,\end{aligned}$$

which implies that $\dot{u} \in L^2(\mathbb{R}^+)$.

Similarly, we also obtain that $G(u) \in L^2(\mathbb{R}^+)$. The proof is complete. □

Theorem 4. *Under the hypotheses of Theorem 3, then $\nabla_g F(\varphi) = 0$ for any $\varphi \in \omega[u]$.*

Proof. Since u is a bounded solution of equation (1), so the ω-limit set $\omega[u]$ is non empty.

Let $\varphi \in \omega[u]$, there exists a sequence $(t_m)_{m\in\mathbb{N}}$ such that $t_m \uparrow \infty$ and $u(t_m)$ tends to φ as m goes to infinity.

By Theorem 3, we have

$$\int_t^{t+1} \|\dot{u}(s)\|^2\mathrm{d}s \to 0 \text{ as } t \to \infty.$$

Thanks to Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}&\sup_{s\in[0,1]}\left\|\int_{t_m}^{t_m+s}\dot{u}(r)\mathrm{d}r\right\| \\ &\qquad\leq \sup_{s\in[0,1]}\int_{t_m}^{t_m+s}\|\dot{u}(r)\|\mathrm{d}r \\ &\qquad= \int_{t_m}^{t_m+s}\|\dot{u}(r)\|\mathrm{d}r \\ &\qquad\leq \left(\int_{t_m}^{t_m+s}\|\dot{u}(r)\|^2\mathrm{d}r\right)^{1/2},\end{aligned}$$

which goes to 0 as m goes to infinity. This means

$$u(t_m+s) = u(t_m) + \int_{t_m}^{t_m+s}\dot{u}(r)\mathrm{d}r \rightrightarrows \varphi,$$

where $\rightrightarrows$ denotes the uniformly convergence. Using the continuity of ∇F, we obtain that $\nabla F(u(t_m+s))$ uniformly converges to $\nabla F(\varphi)$ in $[0,1]$.

Moreover, for any $v \in \mathbb{R}^n$, we have

$$\begin{aligned}&\left|\int_{t_m}^{t_m+1}\langle\dot{u}(s),v\rangle_{g(u(s))}\mathrm{d}s\right| \\ &\qquad\leq \int_{t_m}^{t_m+1}|\langle\dot{u}(s),v\rangle_{g(u(s))}|\mathrm{d}s \\ &\qquad\leq \beta^2\int_{t_m}^{t_m+1}\|\dot{u}(s)\|\|v\|\mathrm{d}s \\ &\qquad\leq \beta^2\|v\|\left(\int_{t_m}^{t_m+1}\|\dot{u}(s)\|^2\mathrm{d}s\right)^{1/2},\end{aligned}$$

which tends to 0 as m goes to infinity. Similarly, we also get

$$\begin{aligned}&\left|\int_{t_m}^{t_m+1}\langle\gamma(s)u(s),v\rangle_{g(u(s))}\mathrm{d}s\right| \\ &\qquad\leq \int_{t_m}^{t_m+1}|\langle\gamma(s)u(s),v\rangle_{g(u(s))}|\mathrm{d}s \\ &\qquad\leq \beta^2\int_{t_m}^{t_m+1}\|\gamma(s)u(s)\|\|v\|\mathrm{d}s \\ &\qquad\leq M\beta^2\|v\|\left(\int_{t_m}^{t_m+1}\|\gamma(s)\|^2\mathrm{d}s\right)^{1/2},\end{aligned}$$

and

$$\begin{aligned}&\left|\int_{t_m}^{t_m+1}\langle f(s),v\rangle_{g(u(s))}\mathrm{d}s\right| \\ &\qquad\leq \int_{t_m}^{t_m+1}|\langle f(s),v\rangle_{g(u(s))}|\mathrm{d}s \\ &\qquad\leq \beta^2\int_{t_m}^{t_m+1}\|f(s)\|\|v\|\mathrm{d}s \\ &\qquad\leq M\beta^2\|v\|\left(\int_{t_m}^{t_m+1}\|f(s)\|^2\mathrm{d}s\right)^{1/2},\end{aligned}$$

where M is an upper bound of the bounded solution u. So we can conclude that the right hand sides vanish as m goes to infinity.

Combining these estimations and equation (1), it follows that

$$\begin{aligned}\int_{t_m}^{t_m+1} &\langle G(u(s)), v\rangle_{g(u(s))} \mathrm{d}s \\ &= -\int_{t_m}^{t_m+1} \langle \dot{u}(s), v\rangle_{g(u(s))} \mathrm{d}s \\ &\quad - \int_{t_m}^{t_m+1} \langle \gamma(s)u(s), v\rangle_{g(u(s))} \mathrm{d}s \\ &\quad + \int_{t_m}^{t_m+1} \langle f(s), v\rangle_{g(u(s))} \mathrm{d}s,\end{aligned}$$

which tends to 0 as m goes to infinity.

Finally, to finish the proof, we can present the inner product of $\nabla F(\varphi)$ and v for any $v \in \mathbb{R}^n$ as follows

$$\begin{aligned}\langle \nabla F(\varphi), v\rangle &= \int_0^1 \langle \nabla F(\varphi), v\rangle \mathrm{d}s \\ &= \lim_{m\to\infty} \int_0^1 \langle \nabla F(u(t_m+s)), v\rangle \mathrm{d}s \\ &= \lim_{m\to\infty} \int_{t_m}^{t_m+1} \langle \nabla F(u(s)), v\rangle \mathrm{d}s \\ &= \lim_{m\to\infty} \int_{t_m}^{t_m+1} \langle \nabla_{g(u)} F(u(s)), v\rangle_{g(u(s))} \mathrm{d}s \\ &= \lim_{m\to\infty} \int_{t_m}^{t_m+1} \langle G(u(s)), v\rangle_{g(u(s))} \mathrm{d}s \\ &= 0.\end{aligned}$$

This equality shows that $\nabla_g F(\varphi) = 0$. □

Theorem 5. *Assume that*

i) the angle and comparability condition (5) of G and ∇F holds;

ii) F satisfies the Kurdyka-Lojasiewicz inequality (7) and the function Θ in (6)-(7) satisfies

$$\Theta(x+y) \le k\Theta(x) + |y|^{1/2}, \ \forall x, y \in \mathbb{R}, \tag{11}$$

for some positive constant k;

iii) $\gamma \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$ such that

$$\int_0^\infty \left(\int_t^\infty \gamma^2(s) ds\right)^{1/2} dt < \infty; \tag{12}$$

iv) $f \in L^1(\mathbb{R}^+, \mathbb{R}^n) \cap L^2(\mathbb{R}^+, \mathbb{R}^n)$ such that

$$\int_0^\infty \left(\int_t^\infty \|f(s)\|^2 ds\right)^{1/2} dt < \infty. \tag{13}$$

If u is a bounded solution to equation (1) then $\dot{u} \in L^1(\mathbb{R}^+)$ and $u(t)$ converges to an equilibrium point $\varphi \in \omega[u]$ at infinity.

Proof. We consider again the energy function Φ defined by (10). It is similar to the proof of Theorem 3, we also get the following estimation

$$-\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) = \frac{1}{2}\left(\|\dot{u}\|_g^2 + \|G(u(t))\|_g^2\right), \tag{14}$$

for every $t \in \mathbb{R}^+$. It implies that the function Φ is non increasing. Moreover, since u is a bounded solution, the ω-limit set $\omega[u]$ is non empty. Let $\varphi \in \omega[u]$, we have $\Phi(t)$ converges to $F(\varphi)$ at infinity. We remark that by subtracting $F(\varphi)$ if needed we may assume $F(\varphi) = 0$. It implies that the energy function $\Phi(t)$ is non negative for every $t \in [0, \infty)$.

It is easily to see that if $\Phi(T) = 0$ for some $T \ge 0$ then u is constant for all $t \ge T$. There remain nothing to prove in this case. Hence, we now assume that $\Phi(t) > 0$ for every $t \in [0, \infty)$.

Since the solution u is bounded, so there exists a positive constant M such that

$$\|u(t)\| \le M, \ \forall t \in [0, \infty).$$

By assumption ii), there exist $\delta > 0$ and the function Θ satisfying (6) such that for all $u \in B_\delta(\varphi)$, there holds

$$\Theta(|F(u)|) \le \|\nabla F(u)\|.$$

We now consider a function I defined by

$$I(x) = \int_0^x \frac{1}{\Theta(s)} \mathrm{d}s, \quad x \in [0, \infty).$$

For any $\varepsilon \in (0, \delta)$, by the hypotheses (12) and (13), there exists t^* such that

$$\|u(t^*) - \varphi\| + \frac{1}{a\alpha^2} I(\Phi(t^*)) < \varepsilon/4,$$

$$\frac{a\beta M}{k} \int_{t^*}^\infty \left(\int_s^\infty |\gamma(r)|^2 \mathrm{d}r\right)^{1/2} \mathrm{d}s < \varepsilon/4,$$

$$\frac{a\beta}{k}\int_{t^*}^{\infty}\left(\int_s^{\infty}\|f(r)\|^2\mathrm{d}r\right)^{1/2}\mathrm{d}s<\varepsilon/4,$$

$$M\int_{t^*}^{\infty}|\gamma(s)|\mathrm{d}s+\int_{t^*}^{\infty}\|f(s)\|\mathrm{d}s<\varepsilon/4.$$

Let us define

$$T=\inf\{t\geq t^*:\ \|u(t)-\varphi\|\geq\varepsilon\}.$$

For every $t\in[t^*,T)$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}I(\Phi(t))=\Phi'(t)\Theta^{-1}(\Phi(t)). \tag{15}$$

Using the definition of energy function Φ in (10) and the hypothesis (11), we obtain

$$\Theta(\Phi(t))\leq k\Theta(F(u(t)))+\frac{1}{\sqrt{2}}\left(\int_t^{\infty}\|\gamma(s)u(s)-f(s)\|^2_{g(u(s))}\mathrm{d}s\right)^{1/2}.$$

According to Kurdyka-Lojasiewicz (7) and the equivalence between Riemannian metric g and Euclidean metric in (9), we have

$$\Theta(\Phi(t))\leq k\|\nabla F(u(t))\|+\frac{\beta}{\sqrt{2}}\left(\int_t^{\infty}\|\gamma(s)u(s)-f(s)\|^2\mathrm{d}s\right)^{1/2}.$$

Applying the angle and comparability condition (5), we get that

$$\Theta(\Phi(t))\leq ka^{-1}\|G(u(t))\|+A, \tag{16}$$

where

$$A=\frac{\beta}{\sqrt{2}}\left(\int_t^{\infty}\|\gamma(s)u(s)-f(s)\|^2\mathrm{d}s\right)^{1/2}.$$

Combining (16) with (15), (14) and (9), it deduces that

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}t}I(\Phi(t))&\geq\frac{\alpha^2k\|\dot u(t)\|\|G(u(t))\|}{ka^{-1}\|G(u(t))\|+A}\\&=a\alpha^2\|\dot u(t)\|-\frac{a\alpha^2A\|\dot u(t)\|}{ka^{-1}\|G(u(t))\|+A}.\end{aligned}$$

It yields that

$$\|\dot u(t)\|\leq-\frac{1}{a\alpha^2}\frac{\mathrm{d}}{\mathrm{d}t}I(\Phi(t))+\frac{A\|\dot u(t)\|}{ka^{-1}\|G(u(t))\|+A}. \tag{17}$$

By equation (1), we obtain

$$\|\dot u(t)\|\leq\|G(u(t)\|+\|\gamma(t)u(t)-f(t)\|.$$

This gives

$$\frac{A\|\dot u(t)\|}{ka^{-1}\|G(u(t))\|+A}\leq\frac{a}{k}A+\|\gamma(t)u(t)-f(t)\|.$$

In the other hand, one has

$$\|\gamma(t)u(t)-f(t)\|\leq M|\gamma(t)|+\|f(t)\|,$$

and

$$A\leq\beta M\left(\int_t^{\infty}|\gamma(s)|^2\mathrm{d}s\right)^{1/2}+\beta\left(\int_t^{\infty}\|f(s)\|^2\mathrm{d}s\right)^{1/2}.$$

Integrating (17) on $[t^*,t)$ for any $t\in[t^*,T)$ we get that

$$\begin{aligned}\int_{t^*}^{t}\|\dot u(s)\|\mathrm{d}s&\leq\frac{1}{a\alpha^2}I(\Phi(t^*))\\&+\frac{a\beta M}{k}\int_{t^*}^{t}\left(\int_s^{\infty}|\gamma(r)|^2\mathrm{d}r\right)^{1/2}\mathrm{d}s\\&+\frac{a\beta}{k}\int_{t^*}^{t}\left(\int_s^{\infty}\|f(r)\|^2\mathrm{d}r\right)^{1/2}\mathrm{d}s\\&+M\int_{t^*}^{t}|\gamma(s)|\mathrm{d}s+\int_{t^*}^{t}\|f(s)\|\mathrm{d}s\\&\leq\frac{1}{a\alpha^2}I(\Phi(t^*))+\frac{3\varepsilon}{4}.\end{aligned}$$

For every $t\in[t^*,T)$, we have

$$\begin{aligned}\|u(t)-\varphi\|&\leq\|u(t)-u(t^*)\|+\|u(t^*)-\varphi\|\\&\leq\int_{t^*}^{T}\|\dot u(s)\|\mathrm{d}s+\|u(t^*)-\varphi\|\\&\leq\frac{1}{a\alpha^2}I(\Phi(t^*))+\|u(t^*)-\varphi\|+\frac{3\varepsilon}{4}\\&<\varepsilon,\end{aligned}$$

where the second estimation is obtained by integrating (17) on $[t^*,T)$. This inequality implies that $T=\infty$ and also yields $\dot u\in L^1(\mathbb{R}^+)$. It follows that the bounded solution $u(t)$ converges to equilibrium point φ at infinity. □

4. Conclusion

In this article, we prove the convergence to equilibrium of bounded solutions to a first order gradient-like system. Our results are obtained under the affect of the low damping and perturbation terms to the asymptotic behavior of solutions at infinity.

Acknowledgment

The second author was supported by Ho Chi Minh City University of Education under grant B2017-SPS-12.

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