An Introductory Overview of Fractional-Calculus Operators Based Upon the Fox-Wright and Related Higher Transcendental Functions

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Abstract. *This survey-cum-expository review article is motivated essentially by the widespread usages of the operators of fractional calculus (that is, fractional-order integrals and fractional-order derivatives) in the modeling and analysis of a remarkably large variety of applied scientific and real-world problems in mathematical, physical, biological, engineering and statistical sciences, and in other scientific disciplines. Here, in this article, we present a brief introductory overview of the theory and applications of the fractional-calculus operators which are based upon the general Fox-Wright function and its such specialized forms as (for example) the widely- and extensively-investigated and potentially useful Mittag-Leffler type functions.*

Keywords

Riemann-Liouville and related fractional derivative operators, Liouville-Caputo fractional derivative operator, hypergeometric functions; special (or higher transcendental) functions; Fox-Wright hypergeometric function; Mittag-Leffler type functions; general Fox-Wright function; zeta and related functions; Lerch transcendent (or Hurwitz-Lerch zeta function); geometric function theory of complex analysis; quantum or basic (***or*** $\mathfrak{q}$-) ***analysis; fractional-order quantum or basic*** (***or*** $\mathfrak{q}$-) ***analysis.***

1. Introduction and motivation

The idea of *fractional calculus* (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has apparently and essentially stemmed from a question raised in the year 1695 by Marquis de l'Hôpital (1661–1704) to

Gottfried Wilhelm Leibniz (1646–1716), which sought the meaning of Leibniz's (currently popular) notation

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$$

for the derivative of order $n \in \mathbb{N}_0 := \{0, 1, 2, \cdots\}$ when $n = \frac{1}{2}$ (What if $n = \frac{1}{2}$?). In his reply, dated 30 September 1695, Leibniz wrote to l'Hôpital as follows:

> "$\cdots$ *This is an apparent paradox from which, one day, useful consequences will be drawn.* $\cdots$"

In recent years, the subject of fractional calculus, as a calculus of integrals and derivatives of any real or complex order, has gained considerable popularity and importance, which is due mainly to its demonstrated applications in the modeling and analysis of applied problems and real-world situations occurring in numerous seemingly diverse and widespread fields of science and engineering. It does indeed also provide several potentially useful tools and techniques for solving differential and integral equations, and various other problems involving special functions of mathematical physics as well as their extensions and generalizations in one and more variables. In a wide variety of applications of fractional calculus, one requires fractional derivatives of different (and, occasionally, *ad hoc*) kinds (see, for example, [42] to [47], [68], [74], [94], [95], [115], [126], [143], [147] and [148]). Traditionally, fractional-order differentiation and integration are defined by the right-sided Riemann-Liouville fractional integral operator ${}^{\mathrm{RL}}I^{\mu}_{a+}$ and the left-sided Riemann-Liouville fractional integral operator ${}^{\mathrm{RL}}I^{\mu}_{a-}$, and the corresponding Riemann-Liouville fractional derivative operators ${}^{\mathrm{RL}}D^{\mu}_{a+}$ and ${}^{\mathrm{RL}}D^{\mu}_{a-}$, as follows (see, for example, [30, Chapter 13], [54, pp. 69–70] and [93]):

$$\left({}^{\mathrm{RL}}I^{\mu}_{a+}f\right)(x) = \frac{1}{\Gamma(\mu)}\int_a^x (x-t)^{\mu-1} f(t)\mathrm{d}t \quad (1)$$
$$\left(x > a;\ \Re(\mu) > 0\right),$$

$$\left({}^{\mathrm{RL}}I^{\mu}_{a-}f\right)(x) = \frac{1}{\Gamma(\mu)}\int_x^a (t-x)^{\mu-1} f(t)\mathrm{d}t \quad (2)$$
$$\left(x < a;\ \Re(\mu) > 0\right)$$

and

$$\left({}^{\mathrm{RL}}D^{\mu}_{a\pm}f\right)(x) = \left(\pm\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(I^{n-\mu}_{a\pm}f\right)(x) \quad (3)$$
$$\left(\Re(\mu) \geqq 0;\ n = [\Re(\mu)] + 1\right),$$

where the function f is locally integrable, $\Re(\mu)$ denotes the real part of the complex number $\mu \in \mathbb{C}$ and $[\Re(\mu)]$ means the greatest integer in $\Re(\mu)$, and $\Gamma(z)$ denotes the classical (Euler's) Gamma function defined by

$$\Gamma(z) := \begin{cases} \displaystyle\int_0^\infty e^{-t}\, t^{z-1}\,\mathrm{d}t, & \left(\Re(z) > 0\right) \\ \displaystyle\frac{\Gamma(z+n)}{\prod_{j=0}^{n-1}(z+j)}, & \left(z \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ n \in \mathbb{N}\right), \end{cases} \quad (4)$$

which happens to be one of the most fundamental and the most useful special functions of mathematical analysis, $\mathbb{N}$ and $\mathbb{Z}_0^-$ being the sets of *positive* and *non-positive* integers, respectively.

An interesting family of generalized Riemann-Liouville fractional derivatives of order μ $(0 < \mu < 1)$ and type ν $(0 \leqq \nu \leqq 1)$ were introduced recently as follows (see [42], [43] and [44]; see also [46], [47] and [94]).

Definition 1. The right-sided Hilfer fractional derivative ${}^{\mathrm{H}}D^{\mu,\nu}_{a+}$ and the left-sided Hilfer fractional derivative ${}^{\mathrm{H}}D^{\alpha,\beta}_{a-}$ of order μ $(0 < \mu < 1)$ and type ν $(0 \leqq \beta \leqq 1)$ with respect to x are defined by

$$\left({}^{\mathrm{H}}D^{\mu,\nu}_{a\pm}f\right)(x)$$
$$= \left(\pm\,{}^{\mathrm{H}}I^{\nu(1-\mu)}_{a\pm}\frac{\mathrm{d}}{\mathrm{d}x}\left({}^{\mathrm{H}}I^{(1-\nu)(1-\mu)}_{a\pm}f\right)\right)(x), \quad (5)$$

where it is tacitly assumed that the second member of (5) exists. The generalization (5) yields the classical Riemann-Liouville fractional derivative operator when $\nu = 0$. Moreover, for $\nu = 1$, it leads to the fractional derivative operator introduced by Liouville [64, p. 10], which is quite frequently attributed to Caputo [23], but which should more appropriately be referred to as the *Liouville-Caputo fractional derivative*, giving due credits to Joseph Liouville (1809–1882) who considered such fractional derivatives many decades earlier in 1832 (see [64]). Many authors (see, for example, [68] and [147]) called the general operators in (5) the Hilfer fractional derivative operators. Several applications of the Hilfer fractional derivative operator $D_{a\pm}^{\alpha,\beta}$ can indeed be found in [44] (see also [103] and [104]).

In this survey-cum-expository review article, our main objective is to present a brief introductory overview of the theory and applications of the fractional-calculus operators which are based upon the general Fox-Wright function and its such specialized forms as the widely- and extensively-investigated and potentially useful Mittag-Leffler type functions.

2. The Fox-Wright function and related Mittag-Leffler type functions

In this section, we begin by introducing the general Fox-Wright function ${}_p\Psi_q \quad (p,q \in \mathbb{N}_0)$ or ${}_p\Psi_q^* \quad (p,q \in \mathbb{N}_0)$, which happens to be the Fox-Wright generalization of the relatively more familiar hypergeometric function ${}_pF_q \quad (p,q \in \mathbb{N}_0)$, with p numerator parameters $a_1, \cdots, a_p$ and q denominator parameters $b_1, \cdots, b_q$ such that

$$a_j \in \mathbb{C} \quad (j = 1, \cdots, p) \text{ and } b_j \in \mathbb{C} \setminus \mathbb{Z}_0^- \quad (j = 1, \cdots, q).$$

All of these specialized higher transcendental functions are, in fact, widely and extensively investigated because mainly of their potential for applications in the mathematical, physical, engineering and statistical sciences.

Definition 2. The general Fox-Wright function ${}_p\Psi_q \quad (p,q \in \mathbb{N}_0)$ or ${}_p\Psi_q^* \quad (p,q \in \mathbb{N}_0)$ is defined by (see, for details, [29, p. 183] and [128, p. 21]; see also [54, p. 56], [51, p. 65] and [125, p. 19])

$$\begin{aligned} {}_p\Psi_q^* \left[\begin{array}{c} (a_1, A_1), \cdots, (a_p, A_p); \\ \\ (b_1, B_1), \cdots, (b_q, B_q); \end{array} z \right] &:= \sum_{n=0}^{\infty} \frac{(a_1)_{A_1 n} \cdots (a_p)_{A_p n}}{(b_1)_{B_1 n} \cdots (b_q)_{B_q n}} \frac{z^n}{n!} \\ &= \frac{\Gamma(b_1) \cdots \Gamma(b_q)}{\Gamma(a_1) \cdots \Gamma(a_p)} \cdot {}_p\Psi_q \left[\begin{array}{c} (a_1, A_1), \cdots, (a_p, A_p); \\ \\ (b_1, B_1), \cdots, (b_q, B_q); \end{array} z \right], \end{aligned} \tag{6}$$

where

$$\Re(A_j) > 0 \quad (j = 1, \cdots, p);$$
$$\Re(B_j) > 0 \quad (j = 1, \cdots, q);$$
$$1 + \Re\left(\sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \right) \geqq 0,$$

and in what follows, $(\lambda)_\nu$ denotes the general Pochhammer symbol or the *shifted* factorial, since

$$(1)_n = n! \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ \mathbb{N} := \{1, 2, 3, \cdots\}),$$

which is defined (for $\lambda, \nu \in \mathbb{C}$ and in terms of the above-defined familiar Gamma function in the equation (4)) by

$$(\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1) \cdots (\lambda+n-1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases} \tag{7}$$

it being assumed *conventionally* that $(0)_0 := 1$ and understood *tacitly* the the Γ-quotient exists. Here we suppose, in general, that

$$a_j, A_j \in \mathbb{C} \quad (j = 1, \cdots, p)$$

and

$$b_j, B_j \in \mathbb{C} \quad (j = 1, \cdots, q)$$

and that the equality in the convergence condition holds true only for suitably bounded values of $|z|$ given by

$$|z| < \nabla := \left(\prod_{j=1}^{p} A_j^{-A_j} \right) \cdot \left(\prod_{j=1}^{q} B_j^{B_j} \right).$$

Clearly, the above-mentioned generalized hypergeoemtric function ${}_pF_q \quad (p,q \in \mathbb{N}_0)$, with p numerator parameters $a_1, \cdots, a_p$ and q denominator parameters $b_1, \cdots, b_q$, is a widely- and extensively-investigated and potentially useful special case of the general Fox-Wright function ${}_p\Psi_q \quad (p, q \in \mathbb{N}_0)$ when

$$A_j = 1 \quad (j = 1, \cdots, p)$$

and

$$B_j = 1 \quad (j = 1, \cdots, q),$$

given by

$$ {}_pF_q \left[\begin{array}{c} a_1, \cdots, a_p; \\ \\ b_1, \cdots, b_q; \end{array} z \right] := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!} $$
$$ = {}_p\Psi_q^* \left[\begin{array}{c} (a_1, 1), \cdots, (a_p, 1); \\ \\ (b_1, 1), \cdots, (b_q, 1); \end{array} z \right] $$
$$ = \frac{\Gamma(b_1) \cdots \Gamma(b_q)}{\Gamma(a_1) \cdots \Gamma(a_p)} $$
$$ \cdot {}_p\Psi_q \left[\begin{array}{c} (a_1, 1), \cdots, (a_p, 1); \\ \\ (b_1, 1), \cdots, (b_q, 1); \end{array} z \right]. \tag{8} $$

We turn now to the familiar Mittag-Leffler function $E_\alpha(z)$ and its two-parameter version $E_{\alpha,\beta}(z)$, which are defined, respectively, by (see [71], [150] and [150])

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \tag{9}$$

$$\big(z, \alpha \in \mathbb{C};\ \Re(\alpha) > 0\big)$$

and

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{10}$$

$$\big(z, \alpha, \beta \in \mathbb{C};\ \Re(\alpha) > 0\big).$$

The one-parameter function $E_\alpha(z)$ was first considered by Magnus Gustaf (Gösta) Mittag-Leffler (1846–1927) in 1903 and its two-parameter version $E_{\alpha,\beta}(z)$ was introduced by Anders Wiman (1865–1959) in 1905 (see also [100]).

The Mittag-Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ are *natural* extensions of the exponential, hyperbolic and trigonometric functions. Indeed, it is easily verified that

$$E_1(z) = e^z,\ E_2\left(z^2\right) = \cosh z, E_2\left(-z^2\right) = \cos z,$$

$$E_{1,2}(z) = \frac{e^z - 1}{z} \quad \text{and} \quad E_{2,2}(z^2) = \frac{\sinh z}{z}.$$

For a reasonably detailed account of the various properties, generalizations and applications of the Mittag-Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$, the reader may refer to the recent works by (for example) Gorenflo *et al.* [36], Haubold *et al.* [40] and Kilbas *et al.* ([52], [53] and [54, Chapter 1]). The Mittag-Leffler function $E_\alpha(z)$ given by (9) and some of its various generalizations have only recently been calculated numerically in the whole complex plane (see, for example, [48] and [96]).

In a remarkably large number of recent investigations, the interest in the families of Mittag-Leffler type functions has grown considerably due mainly to their potential for applications in some reaction-diffusion and other applied sciences and engineering problems. Moreover, their various extensions and generalizations appear in the solutions of fractional-order differential and integral equations (see, for example, [103]; see also [32] and [122]). The following family of the multi-index Mittag-Leffler functions:

$$E_{\gamma,\kappa,\epsilon}\left[(\alpha_j, \beta_j)_{j=1}^m; z\right]$$

was considered and used as a kernel of some fractional-calculus operators by Srivastava *et al.* (see [113] and [114]; see also the references cited in each of these papers):

$$E_{(\alpha_j,\beta_j)_m}^{\gamma,\kappa,\delta,\epsilon}[z] = E_{\gamma,\kappa,\delta,\epsilon}\left[(\alpha_j, \beta_j)_{j=1}^m; z\right] := \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}\,(\delta)_{\epsilon n}}{\prod_{j=1}^{m} \Gamma(\alpha_j n + \beta_j)} \frac{z^n}{n!} \tag{11}$$

$$\left(\alpha_j, \beta_j, \gamma, \kappa, \delta, \epsilon \in \mathbb{C};\ \Re(\alpha_j) > 0 (j = 1, \cdots, m); \right.$$
$$\left. \Re\left(\sum_{j=1}^{m} \alpha_j\right) > \Re(\kappa + \epsilon) - 1\right),$$

where the general Pochhammer symbol $(\lambda)_\nu$ is defined above by (7).

In terms of the general Fox-Wright function ${}_p\Psi_q \quad (p, q \in \mathbb{N}_0)$ or ${}_p\Psi_q^* \quad (p, q \in \mathbb{N}_0)$, which is given by (6), it is easy to observe from the definition (11) that

$$E^{\gamma,\kappa,\delta,\epsilon}_{(\alpha_j,\beta_j)_m}[z] = E_{\gamma,\kappa,\delta,\epsilon}\left[(\alpha_j,\beta_j)_{j=1}^m; z\right] = \frac{1}{\Gamma(\beta_1)\cdots\Gamma(\beta_m)} \cdot {}_2\Psi^*_m\left[\begin{matrix}(\gamma,\kappa),(\delta,\epsilon);\\ (\beta_1,\alpha_1),\cdots,(\beta_m,\alpha_m);\end{matrix}\; z\right] = \frac{1}{\Gamma(\gamma)\Gamma(\delta)}\,{}_2\Psi_m\left[\begin{matrix}(\gamma,\kappa),(\delta,\epsilon);\\ (\beta_1,\alpha_1),\cdots,(\beta_m,\alpha_m);\end{matrix}\; z\right] \tag{12}$$

under the parameter and argument constraints which would correspond appropriately to those that are already listed with the definitions (6) and (11).

We list below some of the special cases of the multi-index Mittag-Leffler function:

$$E_{\gamma,\kappa,\epsilon}\left[(\alpha_j,\beta_j)_{j=1}^m; z\right],$$

which include (for example) the following extensions and generalizations of the Mittag-Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$:

(i) In light of the relation between the Gamma function and the Pochhammer symbol in (7), the case when $m = 2$, $\delta = \epsilon = 1$, $\kappa = \mathfrak{q}$, $\alpha_1 = \alpha$, $\beta_1 = \beta$, and $\alpha_2 = \mathfrak{p}$, and $\beta_2 = \delta$, the definition (11) would correspond to the following relationship:

$$E^{\gamma,\delta,\mathfrak{q}}_{\alpha,\beta,\mathfrak{p}}(z) := \sum_{n=0}^{\infty} \frac{(\gamma)_{\mathfrak{q}n}}{\Gamma(\alpha n+\beta)}\frac{z^n}{(\delta)_{\mathfrak{p}n}} = \frac{1}{\Gamma(\beta)}\,{}_2\Psi^*_2\left[\begin{matrix}(\gamma,\mathfrak{q}),(1,1);\\ (\beta,\alpha),(\delta,\mathfrak{p});\end{matrix}\; z\right] = \frac{\Gamma(\delta)}{\Gamma(\gamma)}\,{}_2\Psi_2\left[\begin{matrix}(\gamma,\mathfrak{q}),(1,1);\\ (\beta,\alpha),(\delta,\mathfrak{p});\end{matrix}\; z\right]$$

with the Mittag-Leffler type function $E^{\gamma,\delta,\mathfrak{q}}_{\alpha,\beta,\mathfrak{p}}(z)$, which was considered by Salim and Faraj [92].

(ii) A special case of the multi-index Mittag-Leffler function defined by (11) when $m = 2$ can be shown to correspond to the Mittag-Leffler function $E^{\gamma,\kappa}_{\alpha,\beta}(z)$:

$$E^{\gamma,\kappa}_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!} = \frac{1}{\Gamma(\beta)}\,{}_1\Psi^*_1\left[\begin{matrix}(\gamma,\kappa);\\ (\beta,\alpha);\end{matrix}\; z\right] = \frac{1}{\Gamma(\gamma)}\,{}_1\Psi_1\left[\begin{matrix}(\gamma,\kappa);\\ (\beta,\alpha);\end{matrix}\; z\right],$$

which was introduced by Srivastava and Tomovski [147] (see also [148]).

(iii) For $m = 2$ and $\kappa = 1$, the multi-index Mittag-Leffler function defined by (11) would correspond readily to the Mittag-Leffler type function $E^{\gamma}_{\alpha,\beta}(z)$:

$$E^{\gamma}_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{(\gamma)_{n}}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!} = \frac{1}{\Gamma(\beta)}\,{}_1\Psi^*_1\left[\begin{matrix}(\gamma,1);\\ (\beta,\alpha);\end{matrix}\; z\right] = \frac{1}{\Gamma(\gamma)}\,{}_1\Psi_1\left[\begin{matrix}(\gamma,1);\\ (\beta,\alpha);\end{matrix}\; z\right], \tag{13}$$

which was studied by Prabhakar [84].

For a large number of other Mittag-Leffler type functions, which are essentially contained in (or analogous to) the general Fox-Wight function $\Psi^*(z)$ or $\Psi(z)$ defined by (11), the interested reader should be referred to the recent works [103], [113] and [114] (see also [142]).

Various special higher transcendental functions of the Mittag-Leffler and the Fox-Wright types are known to play an important rôle in the theory of fractional and operational calculus and their applications in the basic processes of evolution, relaxation, diffusion, oscillation, and wave propagation. Just as we have remarked above, the Mittag-Leffler type functions have only recently been calculated numerically in the whole complex plane (see, for example, [48] and [96]; see also [1] and [78]). Furthermore, several general families of Mittag-Leffler type functions were investigated and applied recently by Srivastava and Tomovski [147]).

In a series of monumental works (see, for example, [152], [153] and [154]), Sir Edward Maitland Wright (1906–2005), with whom I had the privilege to meet and discuss researches emerging from his publications on hypergeometric and related functions during my visit to the University of Aberdeen in the year 1976, introduced and systematically studied the asymptotic expansion of the following Taylor-Maclaurin series (see [152, p. 424]):

$$\mathfrak{E}_{\alpha,\beta}(\phi;z) := \sum_{n=0}^{\infty} \frac{\phi(n)}{\Gamma(\alpha n+\beta)}\, z^n \qquad (14)$$
$$\big(\alpha,\beta \in \mathbb{C};\ \Re(\alpha)>0\big),$$

where $\phi(t)$ is a function satisfying suitable conditions. Wright's above-cited papers were motivated essentially by the earlier developments reported for simpler cases by Magnus Gustaf (Gösta) Mittag-Leffler (1846–1927) in 1905, Anders Wiman (1865–1959) in 1905, Ernest William Barnes (1874–1953) in 1906, Godfrey Harold Hardy (1877–1947) in 1905, George Neville Watson (1886–1965) in 1913, Charles Fox (1897–1977) in 1928, and other authors. In particular, the aforementioned work [19] by *Bishop* Ernest William Barnes (1874–1953) of the Church of England in Birmingham considered the asymptotic expansions of functions in the class defined below:

$$E_{\alpha,\beta}^{(\kappa)}(s;z) := \sum_{n=0}^{\infty} \frac{z^n}{(n+\kappa)^s\,\Gamma(\alpha n+\beta)} \qquad (15)$$
$$\big(\alpha,\beta \in \mathbb{C};\ \Re(\alpha)>0\big)$$

for suitably-restricted parameters κ and s. It is easy to deduce, from the definition (15), the following relationships with the Mittag-Leffler type function $E_{\alpha,\beta}^{(\kappa)}(s;z)$ of Barnes [19]:

$$E_{\alpha}(z) = \lim_{s\to 0}\left\{E_{\alpha,1}^{(\kappa)}(s;z)\right\} \qquad (16)$$

and

$$E_{\alpha,\beta}(z) = \lim_{s\to 0}\left\{E_{\alpha,\beta}^{(\kappa)}(s;z)\right\}. \qquad (17)$$

More interestingly, we also have the following relationship:

$$\lim_{\alpha\to 0}\left\{E_{\alpha,\beta}^{(\kappa)}(s;z)\right\} = \frac{1}{\Gamma(\beta)}\,\Phi(z,s,\kappa)$$

with the classical Lerch transcendent (or the Hurwitz-Lerch zeta function) $\Phi(z,s,\kappa)$ defined by (see, for example, [29, p. 27, Eq. 1.11 (1)]; see also [118, p. 121, *et seq.*])

$$\Phi(z,s,\kappa) := \sum_{n=0}^{\infty} \frac{z^n}{(n+\kappa)^s} \qquad (18)$$

$\big(\kappa \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ s\in\mathbb{C}$ when $|z|<1$; $\Re(s)>1$ when $|z|=1\big)$.

The Hurwitz-Lerch zeta function $\Phi(z,s,\kappa)$ defined by (18) contains, as its *special* cases, not only the Riemann zeta function $\zeta(s)$ and the Hurwitz (or generalized) zeta function $\zeta(s,\kappa)$:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \Phi(1,s,1), \qquad (19)$$

$$\zeta(s,\kappa) := \sum_{n=0}^{\infty} \frac{1}{(n+\kappa)^s} = \Phi(1,s,\kappa) \qquad (20)$$

and the Lerch zeta function $\ell_s(\xi)$ defined by (see, for details, [29, Chapter I] and [118, Chapter 2])

$$\ell_s(\xi) := \sum_{n=1}^{\infty} \frac{e^{2n\pi i\xi}}{n^s} = e^{2\pi i\xi}\,\Phi\left(e^{2\pi i\xi},s,1\right) \qquad (21)$$

$$\big(\mathrm{i}=\sqrt{-1};\ \xi\in\mathbb{R};\ \Re(s)>1\big),$$

but also such other important functions of *Analytic Number Theory* as the Polylogarithmic function (or *de Jonquière's function*) $\mathrm{Li}_s(z)$:

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\,\Phi(z,s,1) \qquad (22)$$

$$\big(s\in\mathbb{C} \text{ when } |z|<1;\ \Re(s)>1 \text{ when } |z|=1\big)$$

and the Lipschitz-Lerch zeta function (see [118, p. 122, Eq. 2.5 (11)]):

$$\phi(\xi,\kappa,s) := \sum_{n=0}^{\infty} \frac{e^{2n\pi i\xi}}{(n+\kappa)^s} = \Phi\left(e^{2\pi i\xi},s,\kappa\right) =: L(\xi,s,\kappa) \qquad (23)$$

$$\big(\kappa \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \Re(s) > 0 \quad \text{when} \quad \xi \in \mathbb{R} \setminus \mathbb{Z};\ \Re(s) > 1 \quad \text{when} \quad \xi \in \mathbb{Z}\big),$$

which was first studied by Rudolf Lipschitz (1832–1903) and Matyáš Lerch (1860–1922) in connection with Dirichlet's famous theorem on primes in arithmetic progressions (see, for details, [105] and [106]).

Asymptotic expansions of such functions as those in the class of the Mittag-Leffler type function $E_{\alpha,\beta}^{(\kappa)}(s;z)$ defined by (15), and the classical Mittag-Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ defined by (9), were discussed by Barnes [19], as we have indicated above. Moreover, as already pointed out categorically and repeatedly in many subsequent publications including (for example) the one by Srivastava *et al.* [144, p. 503, Eq. (6.3)] for the following "generalized" M-series introduced recently by Sharma and Jain [97] by

$$\begin{aligned} &{}_pM_q^{\alpha,\beta}(a_1,\cdots,a_p;b_1,\cdots,b_q;z) \\ &= \sum_{k=0}^{\infty} \frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\,\frac{z^k}{\Gamma(\alpha k+\beta)} \\ &= \frac{1}{\Gamma(\beta)} \cdot {}_{p+1}\Psi_{q+1}^{*}\left[\begin{matrix} (a_1,1),\cdots,(a_p,1),(1,1); \\ (b_1,1),\cdots,(b_q,1),(\beta,\alpha); \end{matrix}\ z\right] \\ &= \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)} \cdot {}_{p+1}\Psi_{q+1}\left[\begin{matrix} (a_1,1),\cdots,(a_p,1),(1,1); \\ (b_1,1),\cdots,(b_q,1),(\beta,\alpha); \end{matrix}\ z\right] \end{aligned} \tag{24}$$

the last relationship in (24) exhibits the fact that the so-called generalized M-series is, in fact, an *obvious* (rather trivial) variant of the Fox-Wright function ${}_p\Psi_q^*$ defined by (6).

A natural unification and generalization of the Fox-Wright function ${}_p\Psi_q^*$ defined by (6) as well as the Hurwitz-Lerch zeta function $\Phi(z,s,\kappa)$ defined by (18) was indeed accomplished by introducing essentially arbitrary numbers of numerator and denominator parameters in the definition (18). For this purpose, in addition to the symbol ∇^* defined by

$$\nabla^* := \left(\prod_{j=1}^{p} \rho_j^{-\rho_j}\right) \cdot \left(\prod_{j=1}^{q} \sigma_j^{\sigma_j}\right), \tag{25}$$

the following notations will be employed:

$$\Delta := \sum_{j=1}^{q} \sigma_j - \sum_{j=1}^{p} \rho_j \tag{26}$$

and

$$\Xi := s + \sum_{j=1}^{q} \mu_j - \sum_{j=1}^{p} \lambda_j + \frac{p-q}{2}. \tag{27}$$

Then the extended Hurwitz-Lerch zeta function

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}(z,s,\kappa)$$

is defined by [144, p. 503, Equation (6.2)] (see also [101] and [119])

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z,s,\kappa) := \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(\lambda_j)_{n\rho_j}}{n!\cdot\prod_{j=1}^{q}(\mu_j)_{n\sigma_j}}\,\frac{z^n}{(n+\kappa)^s} \tag{28}$$

$$\Big(p,q \in \mathbb{N}_0;\ \lambda_j \in \mathbb{C}\ (j=1,\cdots,p);\ \kappa,\mu_j \in \mathbb{C}\setminus Z_0^-\ (j=1,\cdots,q);\ \rho_j,\sigma_k \in \mathbb{R}^+\ (j=1,\cdots,p;\ k=1,\cdots,q);\ \Delta > -1 \ \text{when}\ s,z\in\mathbb{C};$$
$$\Delta = -1 \ \text{and}\ s \in \mathbb{C} \ \text{when}\ |z| < \nabla^*;$$
$$\Delta = -1 \ \text{and}\ \Re(\Xi) > \frac{1}{2} \ \text{when}\ |z| = \nabla^*\Big).$$

For an interesting and potentially useful family of λ-generalized Hurwitz-Lerch zeta functions, which *further* extend the multi-parameter Hurwitz-Lerch zeta function

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}(z,s,\kappa)$$

defined by (28), was introduced and investigated systematically in a recent paper by Srivastava [102], who also discussed their potential application in Number Theory by appropriately constructing a presumably new continuous analogue of Lippert's Hurwitz measure and also considered some other statistical applications of these families of the λ-generalized Hurwitz-Lerch zeta functions in probability distribution theory (see also the references to several related earlier works cited by Srivastava [102]).

Remark 1. If we set

$$s = 0, \quad p \mapsto p+1$$
$$(\rho_1 = \cdots = \rho_p = 1; \quad \lambda_{p+1} = \rho_{p+1} = 1)$$

and

$$q \mapsto q+1$$
$$(\sigma_1 = \cdots = \sigma_q = 1; \quad \mu_{q+1} = \beta; \quad \sigma_{q+1} = \alpha),$$

then (28) reduces immediately to the M-series in (24).

Remark 2. If, in Wright's definition (14) of 1940 in [152], we set $\alpha = \beta = 1$ and

$$\phi(n) = \frac{\prod_{j=1}^{p} \Gamma(a_j + A_j n)}{\prod_{j=1}^{q} \Gamma(b_j + B_j n)} \qquad (n \in \mathbb{N}_0) \tag{29}$$

or, alternatively, if we let $\alpha \to 0$, $\beta = 1$ and

$$\phi(n) = \frac{\prod_{j=1}^{p} \Gamma(a_j + A_j n)}{n! \cdot \prod_{j=1}^{q} \Gamma(b_j + B_j n)} \qquad (n \in \mathbb{N}_0) \tag{30}$$

or, more simply, if we put

$$\phi(n) = \frac{\Gamma(\alpha n + \beta) \prod_{j=1}^{p} \Gamma(a_j + A_j n)}{n! \cdot \prod_{j=1}^{q} \Gamma(b_j + B_j n)} \quad (n \in \mathbb{N}_0), \tag{31}$$

then (14) would immediately yield the familiar Fox-Wright hypergeometric function ${}_p\Psi_q(z)$ defined by (6).

Finally, in this section, we introduce the following interesting unification of the definitions in (14) and (28) for suitably-restricted function $\varphi(\tau)$:

$$\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa) := \sum_{n=0}^{\infty} \frac{\varphi(n)}{(n+\kappa)^s \, \Gamma(\alpha n + \beta)} \, z^n \tag{32}$$
$$(\alpha, \beta \in \mathbb{C}; \; \Re(\alpha) > 0),$$

where the parameters α, β, s and κ are appropriately constrained as above.

Remark 3. Clearly, if we replace the sequence $\{\varphi(n)\}_{n=0}^{\infty}$ in the definition (32) by the sequence $\{\phi(n)\}_{n=0}^{\infty}$, we have

$$\mathfrak{E}_{\alpha,\beta}(\phi; z) = \lim_{s \to 0} \{\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa)\} \Big|_{\varphi \equiv \phi}. \tag{33}$$

Moreover, if in the definition (32), we set $\alpha = \beta = 1$ and

$$\varphi(n) = \frac{\prod_{j=1}^{p} (\lambda_j)_{n\rho_j}}{\prod_{j=1}^{q} (\mu_j)_{n\sigma_j}} \qquad (n \in \mathbb{N}_0), \tag{34}$$

then the definition (32) will immediately yield the definition (28) of the extended Hurwitz-Lerch zeta function

$$\Phi^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z, s, \kappa).$$

Alternatively, in the special case of (32) when $\alpha \to 0$, $\beta = 1$ and

$$\varphi(n) = \frac{\prod_{j=1}^{p} (\lambda_j)_{n\rho_j}}{n! \cdot \prod_{j=1}^{q} (\mu_j)_{n\sigma_j}} \qquad (n \in \mathbb{N}_0) \tag{35}$$

or, more simply, when we set

$$\varphi(n) = \frac{\Gamma(\alpha n + \beta) \prod_{j=1}^{p} (\lambda_j)_{n\rho_j}}{n! \cdot \prod_{j=1}^{q} (\mu_j)_{n\sigma_j}} \quad (n \in \mathbb{N}_0), \tag{36}$$

we are led to the extended Hurwitz-Lerch zeta function

$$\Phi^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z, s, \kappa)$$

defined by (28).

3. Fractional-Calculus Operators with $\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa)$ as the Kernel

We begin this section by remarking that, not only the Fox-Wright hypergeometric function

${}_p\Psi_q(z)$ defined by (6), but also much more general functions such as (for example) Meijer's G-function and Fox's H-function, have already been used as kernels of many different families of fractional-calculus operators (see, for details, [125]. [126] and [142]; see also the references cited in each of these earlier works). As a matter of fact, Srivastava *et al.* [126] not only used the Riemann-Liouville type fractional integrals with the Fox H-function and the Fox-Wright hypergeometric function ${}_p\Psi_q(z)$ as kernels, but also applied their results to the substantially more general $\overline{H}$-function (see, for example, [22] and [130]).

Wright's function $\mathfrak{E}_{\alpha,\beta}(\varphi; z)$ in (14), which was introduced in [152] in 1940, has appeared recently in [86] in connection with fractional calculus, but without giving due credits to Wright [152]. Here, in this section, we begin by the following general family of operators of fractional integrals and fractional derivatives of the Riemann-Liouville kind, which involve the function $\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa)$ in their kernel.

Definition 3. The general right-sided fractional integral operator $\mathcal{I}^{\mu}_{a+}(\varphi; s, \kappa)$ and the general left-sided fractional integral operator $\mathcal{I}^{\mu}_{a-}(\varphi; z, s, \kappa, \nu)$, and the corresponding fractional derivative operators $\mathcal{D}^{\mu}_{a+}(\varphi; z, s, \kappa, \nu)$ and $\mathcal{D}^{\mu}_{a-}(\varphi; z, s, \kappa, \nu)$, each of the Riemann-Liouville type, are defined by

$$\left(\mathcal{I}^{\mu}_{a+}(\varphi; z, s, \kappa, \nu)f\right)(x) = \int_a^x \frac{(x-t)^{\mu-1}}{\Gamma(\mu)}\, \mathcal{E}_{\alpha,\beta}\big(\varphi; z(x-t)^{\nu}, s, \kappa\big) f(t)\mathrm{d}t \tag{37}$$

$$\big(x > a;\ \Re(\mu) > 0\big),$$

$$\left(\mathcal{I}^{\mu}_{a-}(\varphi; z, s, \kappa, \nu)f\right)(x) = \int_x^a \frac{(t-x)^{\mu-1}}{\Gamma(\mu)}\, \mathcal{E}_{\alpha,\beta}\big(\varphi; z(t-x)^{\nu}, s, \kappa\big) f(t)\mathrm{d}t \tag{38}$$

$$\big(x < a;\ \Re(\mu) > 0\big)$$

and

$$\left(\mathcal{D}^{\mu}_{a\pm}(\varphi; z, s, \kappa, \nu)f\right)(x) = \left(\pm\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\mathcal{I}^{n-\mu}_{a\pm}(\varphi; z, s, \kappa, \nu)f\right)(x) \tag{39}$$

$$\big(\Re(\mu) \geqq 0;\ n = [\Re(\mu)] + 1\big),$$

where the function f is in the space $L(\mathfrak{a}, \mathfrak{b})$ of Lebesgue integrable functions on a finite closed interval $[\mathfrak{a}, \mathfrak{b}]$ $(\mathfrak{b} > \mathfrak{a})$ of the real line $\mathbb{R}$ given by

$$L(\mathfrak{a}, \mathfrak{b}) = \left\{ f:\ \ \|f\|_1 = \int_{\mathfrak{a}}^{\mathfrak{b}} |f(x)|\, \mathrm{d}x < \infty \right\}, \tag{40}$$

it being *tacitly* assumed that, in situations such as those occurring in conjunction with the usages of the definitions in (37), (38) and (39), the point $\mathfrak{a}$ in all such function spaces as (for example) the function space $L(\mathfrak{a}, \mathfrak{b})$ coincides precisely with the *lower* terminal a in the integrals involved in the definitions (37), (38) and (39).

Remark 4. It is easily seen from the definition (32) that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left\{x^{\mu-1}\, \mathcal{E}_{\alpha,\beta}\big(\varphi; zx^{\nu}, s, \kappa\big)\right\} = x^{\mu-n-1} \cdot \sum_{k=0}^{\infty} \frac{\varphi(k)}{(k+\kappa)^s\, \Gamma(\alpha k + \beta)}\, \frac{\Gamma(\nu k + \mu)}{\Gamma(\nu k + \mu - n)}\, (zx^{\nu})^k \tag{41}$$

$$\big(n \in \mathbb{N}_0;\ \Re(\mu) > 0;\ \Re(\nu) > 0;\ \Re(\alpha) > 0\big),$$

which, in the special case when $\mu = \beta$ and $\nu = \alpha$, yields

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left\{x^{\beta-1}\, \mathcal{E}_{\alpha,\beta}\big(\varphi; zx^{\alpha}, s, \kappa\big)\right\} = x^{\beta-n-1}\, \mathcal{E}_{\alpha,\beta-n}\big(\varphi; zx^{\alpha}, s, \kappa\big) \tag{42}$$

$$\big(n \in \mathbb{N}_0;\ \Re(\alpha) > 0;\ \Re(\beta) > 0\big),$$

provided that each member of the equations (41) and (42) exists.

Remark 5. Upon setting

$$\mathfrak{I}\left\{t^{\mu-1}\, \mathcal{E}_{\alpha,\beta}\big(\varphi; zt^{\nu}, s, \kappa\big)\right\}(x) = \int_0^x t^{\mu-1}\, \mathcal{E}_{\alpha,\beta}\big(\varphi; zt^{\nu}, s, \kappa\big)\, \mathrm{d}t, \tag{43}$$

if we make use of term-by-term integration in conjunction with the definition (32), we find that

$$\mathfrak{I}\left\{t^{\mu-1}\, \mathcal{E}_{\alpha,\beta}\big(\varphi; zt^{\nu}, s, \kappa\big)\right\}(x) = x^{\mu} \sum_{k=0}^{\infty} \frac{\varphi(k)}{(k+\kappa)^s\, \Gamma(\alpha k + \beta)} \cdot \frac{\Gamma(\nu k + \mu)}{\Gamma(\nu k + \mu + 1)}\, (zx^{\nu})^k \tag{44}$$

$$\big(\Re(\mu) > 0;\ \Re(\nu) > 0;\ \Re(\alpha) > 0\big).$$

provided that the integral exists. By iterating this process of integration $n-1$ times for $n \in \mathbb{N}$, we are led eventually to the following integral formula:

$$\begin{aligned}\mathfrak{I}^n \left\{t^{\mu-1}\, \mathcal{E}_{\alpha,\beta}\big(\varphi; zt^{\nu}, s, \kappa\big)\right\}(x) \\ = x^{\mu+n-1} \sum_{k=0}^{\infty} \frac{\varphi(k)}{(k+\kappa)^s\, \Gamma(\alpha k+\beta)} \\ \cdot \frac{\Gamma(\nu k+\mu+n-1)}{\Gamma\big(\nu k+\mu+n\big)}\, \big(zx^{\nu}\big)^k \end{aligned} \tag{45}$$

$$\big(n \in \mathbb{N};\ \Re(\mu) > 0;\ \Re(\nu) > 0;\ \Re(\alpha) > 0\big).$$

In particular, when $\mu = \beta$ and $\nu = \alpha$, we find from (45) that

$$\begin{aligned}\mathfrak{I}^n \left\{t^{\beta-1}\, \mathcal{E}_{\alpha,\beta}\big(\varphi; zt^{\alpha}, s, \kappa\big)\right\}(x) \\ = x^{\beta+n-1}\, \mathcal{E}_{\alpha,\beta+n}\big(\varphi; zx^{\alpha}, s, \kappa\big)\end{aligned} \tag{46}$$

$$\big(n \in \mathbb{N};\ \Re(\alpha) > 0;\ \Re(\beta) > 0\big),$$

provided that each member of the equations (45) and (46) exists.

Remark 6. In terms of the operator $\mathcal{L}$ of the Laplace transform given by

$$\mathcal{L}\left\{f(\tau) : \mathfrak{s}\right\} := \int_0^{\infty} \mathrm{e}^{-\mathfrak{s}\tau}\, f(\tau)\, \mathrm{d}\tau =: F(\mathfrak{s}) \tag{47}$$
$$\big(\Re(\mathfrak{s}) > 0\big),$$

where the function $f(\tau)$ is so constrained that the integral exists, it is easily seen for the function $\mathcal{E}_{\alpha,\beta}\big(\varphi; z, s, \kappa\big)$, defined above by (32), that

$$\begin{aligned}\mathcal{L}\left\{\tau^{\mu-1}\, \mathcal{E}_{\alpha,\beta}\big(\varphi; z\tau^{\nu}, s, \kappa\big) : \mathfrak{s}\right\} \\ = \frac{1}{\mathfrak{s}^{\mu}} \sum_{k=0}^{\infty} \frac{\varphi(k)\, \Gamma(\nu k+\mu)}{(k+\kappa)^s\, \Gamma(\alpha k+\beta)} \left(\frac{z}{\mathfrak{s}^{\nu}}\right)^k\end{aligned} \tag{48}$$

$$\big(\Re(\mathfrak{s}) > 0;\ \Re(\mu) > 0;\ \Re(\nu) > 0;\ \Re(\alpha) > 0\big),$$

provided that each member of (48) exists. Obviously, when $\mu = \beta$ and $\nu = \alpha$, the Laplace transform formula (48) simplifies to the following form:

$$\begin{aligned}\mathcal{L}\left\{\tau^{\beta-1}\, \mathcal{E}_{\alpha,\beta}\big(\varphi; z\tau^{\alpha}, s, \kappa\big) : \mathfrak{s}\right\} \\ = \frac{1}{\mathfrak{s}^{\mu}} \sum_{k=0}^{\infty} \frac{\varphi(k)}{(k+\kappa)^s} \left(\frac{z}{\mathfrak{s}^{\alpha}}\right)^k\end{aligned} \tag{49}$$

$$\big(\Re(\mathfrak{s}) > 0;\ \Re(\alpha) > 0;\ \Re(\beta) > 0\big).$$

Remark 7. By appropriately applying the Laplace transform formula (48) in conjunction with the definition (39), it is not difficult to deduce the Laplace transform formula for the generalized fractional derivative of the Riemann-Liouville form given by (39). The generalized fractional derivative of the Liouville-Caputo form involving the general function $\mathcal{E}_{\alpha,\beta}\big(\varphi; z, s, \kappa\big)$, given by (32), can indeed be defined and handled analogously. For the sake of brevity, we choose to leave the details involved in these derivations as an exercise for the interested users of such types of generalized fractional derivatives.

By applying the limit formula (33) or, alternatively, if we make use of the definitions in (14) and (47), we find for Wright's function $\mathfrak{E}_{\alpha,\beta}\big(\phi; z\big)$ that

$$\begin{aligned}\mathcal{L}\left\{\tau^{\mu-1}\, \mathfrak{E}_{\alpha,\beta}\big(\varphi; z\tau^{\nu}\big) : \mathfrak{s}\right\} \\ = \frac{1}{\mathfrak{s}^{\mu}} \sum_{k=0}^{\infty} \frac{\phi(k)\, \Gamma(\nu k+\mu)}{\Gamma(\alpha k+\beta)} \left(\frac{z}{\mathfrak{s}^{\nu}}\right)^k\end{aligned} \tag{50}$$

$$\big(\Re(\mathfrak{s}) > 0;\ \Re(\mu) > 0;\ \Re(\nu) > 0;\ \Re(\alpha) > 0\big),$$

which, in the special case when $\nu = \alpha$ and $\mu = \beta$, yields

$$\begin{aligned}\mathcal{L}\left\{\tau^{\beta-1}\, \mathfrak{E}_{\alpha,\beta}\big(\varphi; z\tau^{\alpha}\big) : \mathfrak{s}\right\} \\ = \frac{1}{\mathfrak{s}^{\beta}} \sum_{k=0}^{\infty} \phi(k) \left(\frac{z}{\mathfrak{s}^{\beta}}\right)^k\end{aligned} \tag{51}$$

$$\big(\Re(\mathfrak{s}) > 0;\ \Re(\alpha) > 0;\ \Re(\beta) > 0\big).$$

Furthermore, if we choose the general sequence $\{\varphi(n)\}_{n=0}^{\infty}$ as follows:

$$\varphi(n) = \frac{(\gamma)_n}{n!}\, (n+\kappa)^s \qquad (n \in \mathbb{N}_0),$$

then this last Laplace transform formula (49) reduces to a known result in the form given by (see [84]):

$$\begin{aligned}\mathcal{L}\left\{\tau^{\beta-1}\, E^{\gamma}_{\alpha,\beta}\big(z\tau^{\alpha}\big) : \mathfrak{s}\right\} = \frac{1}{\mathfrak{s}^{\beta}} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k!} \left(\frac{z}{\mathfrak{s}^{\alpha}}\right)^k \\ = \frac{\mathfrak{s}^{\gamma\alpha-\beta}}{(\mathfrak{s}^{\beta}-z)^{\gamma}}\end{aligned} \tag{52}$$

$$\big(\Re(\mathfrak{s}) > 0;\ \Re(\alpha) > 0;\ \Re(\beta) > 0;\ \big),$$

where the Mittag-Leffler type function $E^{\gamma}_{\alpha,\beta}(z)$ is defined by the equation (13). More generally, if the sequence $\{\varphi(n)\}_{n=0}^{\infty}$ is given by (35), then the Laplace transformation formula (52) would yield the following result:

$$\mathcal{L}\left\{\tau^{\mu-1}\,\Phi^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}\big(z\tau^{\nu},s,\kappa\big):\mathfrak{s}\right\}$$

$$=\frac{\Gamma(\mu)}{\mathfrak{s}^{\mu}}\,\Phi^{(\nu,\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}_{\mu,\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}\left(\frac{z}{\mathfrak{s}^{\nu}},s,\kappa\right) \quad (53)$$

$$\big(\Re(\mathfrak{s})>0;\ \Re(\mu)>0;\ \Re(\nu)>0;\ \Re(\alpha)>0\big)$$

for the extended Hurwitz-Lerch zeta function

$$\Phi^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,\kappa)$$

defined by (28).

In solving various applied problems, which are modeled as initial-value problems for fractional differential equations involving special cases of the fractional-calculus operators given by Definition 3, use is made of the Laplace transform method based upon such Laplace transform formulas as those listed in Remark 6.

4. Fractional-order modeling and analysis of initial-value problems

In this section, we present several examples which would illustrate the fractional-order modeling and analysis of a variety of initial-value problems involving ordinary and partial differential equations. For simplicity and convenience, we consider the case $a = 0$ of the definitions given by the equations (1), (2) and (3) as follows:

$$\left({}^{\mathrm{RL}}I^{\mu}_{0+}f\right)(x)=\int_0^x\frac{(x-t)^{\mu-1}}{\Gamma(\mu)}f(t)\mathrm{d}t \quad (54)$$

$$\big(x>0;\ \Re(\mu)>0\big),$$

$$\left({}^{\mathrm{RL}}I^{\mu}_{0-}f\right)(x)=\int_x^a\frac{(t-x)^{\mu-1}}{\Gamma(\mu)}\,f(t)\,\mathrm{d}t \quad (55)$$

$$\big(x<0;\ \Re(\mu)>0\big)$$

and

$$\left({}^{\mathrm{RL}}D^{\mu}_{0\pm}f\right)(x)=\left(\pm\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n}\left(I^{n-\mu}_{0\pm}f\right)(x) \quad (56)$$

$$\big(\Re(\mu)\geqq 0;\ n=[\Re(\mu)]+1\big),$$

where, as before, the function f is locally integrable, $\Re(\mu)$ denotes the real part of the complex number $\mu\in\mathbb{C}$ and $[\Re(\mu)]$ means the greatest integer in $\Re(\mu)$. Thus, for the Riemann-Liouville fractional derivative operator $\mathcal{D}^{\mu}_{0+}$ of order μ in the definition (56), it is easily seen that

$$\mathcal{L}\left\{\left({}^{\mathrm{RL}}D^{\mu}_{0+}f\right)(t):\mathfrak{s}\right\}$$

$$=\mathfrak{s}^{\mu}\,F(\mathfrak{s})-\sum_{k=0}^{n-1}\mathfrak{s}^{k}\,\left({}^{\mathrm{RL}}D^{\mu-k-1}_{0+}f\right)(t)\Big|_{t=0} \quad (57)$$

$$\big(n-1\leqq\Re(\mu)<n;\ n\in\mathbb{N}\big),$$

where $\mathcal{L}$ is the operator of the Laplace transform given by (47). However, for the *ordinary* derivative $f^{(n)}(t)$ order $n\in\mathbb{N}_0$), it is known that

$$\mathcal{L}\left\{f^{(n)}(t):\mathfrak{s}\right\}$$

$$=\mathfrak{s}^{n}\,F(\mathfrak{s})-\sum_{k=0}^{n-1}\mathfrak{s}^{k}\,f^{(n-k-1)}(t)\Big|_{t=0}\quad(n\in\mathbb{N}_0) \quad (58)$$

or, equivalently, that

$$\mathcal{L}\left\{f^{(n)}(t):\mathfrak{s}\right\}$$

$$=\mathfrak{s}^{n}\,F(\mathfrak{s})-\sum_{k=0}^{n-1}\mathfrak{s}^{n-k-1}\,f^{(k)}(0+)\ (n\in\mathbb{N}_0), \quad (59)$$

where, as well as in all of such situations in this paper, an *empty* sum is to be interpreted as 0.

Clearly, from the Laplace transform formulas (57) and (58), it is observed that the initial values such as those that occur in (57) are usually not interpretable physically in a given initial-value problem. Besides, unfortunately, the Riemann-Liouville fractional derivative of a constant is not zero. These and other situations and disadvantages are overcome at least partially by means of the Liouville-Caputo fractional derivative which, as we indicated in the introductory Section 1, was considered in an earlier work dated 1832 by Joseph Liouville (1809–1882) [64, p. 10] *and* which has arisen in several important recent works, dated 1969 onwards, by Michele Caputo (see, for details, [82, p. 78 *et seq.*]; see also [54, p. 90 *et seq.*]).

In many recent works, especially in the theory of viscoelasticity and in hereditary solid mechanics, the following definition dated 1832 of Liouville [64] and dated 1969 of Caputo [23] is adopted for the fractional derivative of order $\mu > 0$ of a *causal* function $f(t)$, that is,

$$f(t) = 0 \qquad (t < 0),$$

given by

$$\begin{aligned}\frac{\mathrm{d}^\mu}{\mathrm{d}x^\mu}\{f(x)\} &= \left({}^{\mathrm{LC}}D^\mu_{0+}f\right)(x) \\ &:= \begin{cases} f^{(n)}(x) \;\; (\mu = n \in \mathbb{N}_0) \\ \dfrac{1}{\Gamma(n-\mu)}\displaystyle\int_0^x \frac{f^{(n)}(t)}{(x-t)^{\mu-n+1}}\,\mathrm{d}t \\ \left(n-1 < \Re(\mu) < n;\; n \in \mathbb{N}\right), \end{cases}\end{aligned} \tag{60}$$

where

$$n = \begin{cases} [\Re(\mu)] + 1 & (\mu \neq \mathbb{N}_0) \\ \mu & (\mu \in \mathbb{N}_0), \end{cases} \tag{61}$$

$f^{(n)}(t)$ denotes, as before, the usual (ordinary) derivative of $f(t)$ of order n and Γ is the familiar (Euler's) Gamma function.

One can apply the above-introduced notion in order to model and analyze some basic situations in applied mathematical and physical sciences, which are treated by simple, linear, ordinary or partial, differential equations, since [see the equation (57) and the definition in (60)]

$$\begin{aligned}&\mathcal{L}\left\{\left({}^{\mathrm{LC}}D^\mu_{0+}f\right)(x) : \mathfrak{s}\right\} \\ &\quad = \mathfrak{s}^\mu F(\mathfrak{s}) - \sum_{k=0}^{n-1} \mathfrak{s}^{\mu-k-1} f^{(k)}(0+) \\ &\left(n-1 < \alpha \leqq n;\; n \in \mathbb{N}_0\right),\end{aligned} \tag{62}$$

which, just as the Laplace transform formulas (58) or (59), is obviously and practically more suited for initial-value problems than the Laplace transform formula (57) (see, for details, the recent article Gorenflo *et al.* [36], and also the monographs by Podlubny [82] and Kilbas *et al.* [54]).

The following relationship between the Riemann-Liouville fractional derivative ${}^{\mathrm{RL}}D^\mu_{0+}$ and the Liouville-Caputo fractional derivative ${}^{\mathrm{LC}}D^\mu_{0+}$ of order μ is known (see, for example, [54, p. 91, Eq. (2.4.1)] *with* $a = 0$):

$$\begin{aligned}&\left({}^{\mathrm{LC}}D^\mu_{0+}f\right)(x) \\ &\quad = \left({}^{\mathrm{RL}}D^\mu_{0+}\left\{f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}\,t^k\right\}\right)(x),\end{aligned} \tag{63}$$

where n is given by (61). Equivalently, since

$$\left({}^{\mathrm{RL}}D^\mu_{0+}\left\{t^{\lambda-1}\right\}\right)(x) = \frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu)}\,x^{\lambda-\mu-1} \tag{64}$$
$$\left(\Re(\lambda) > 0;\; \Re(\mu) \geqq 0\right),$$

the relationship (63) can be written as follows:

$$\begin{aligned}\left({}^{\mathrm{LC}}D^\mu_{0+}f\right)(x) &= \left({}^{\mathrm{RL}}D^\mu_{0+}f\right)(x) \\ &\quad - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{\Gamma(k-\mu+1)}\,x^{k-\mu},\end{aligned} \tag{65}$$

where n is given, as in (63), by (61).

We give below three examples of how fractional-order derivatives are potentially useful in the modeling and analysis of applied problems.

Example 1. The following first- and second-order linear ordinary differential equations:

$$\frac{\mathrm{d}y}{\mathrm{d}t} + cy = 0 \quad (c > 0),$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + cy = 0 \quad (c > 0)$$

are usually referred to as the *relaxation equation* and the *oscillation equation*, respectively. Also, in the theory of partial differential equations, the following partial differential equations:

$$\frac{\partial^2 u}{\partial x^2} = k\frac{\partial u}{\partial t} \quad (k > 0)$$
$$\frac{\partial^2 u}{\partial x^2} = k\frac{\partial^2 u}{\partial t^2} \quad (k > 0)$$

are known as the *diffusion* (*or heat*) *equation* and the *wave equation*, respectively.

Let us recall that the basic processes of relaxation, diffusion, oscillations and wave propagation have been revisited by several authors by introducing fractional-order derivatives in the governing (ordinary or partial) differential equations. This leads to *superslow* or *intermediate processes* that, in mathematical physics, we may refer to as *fractional phenomena*. The analysis of each of these phenomena, when carried out by means of fractional calculus and Laplace transforms, involves such special functions in one variable as those of the Mittag-Leffler and Fox-Wright types. These useful special functions are investigated systematically as relevant cases of the general class of functions which are popularly known as Fox's H-function after Charles Fox (1897–1977) who initiated a detailed study of these functions as symmetrical Fourier kernels (see, for details, [125] and [128]). As a matter of fact, as we shall see in Section 5 of this article, mathematical modeling and analysis of real-world and other applied problems are being accomplished widely and extensively by making use of fractional-order derivatives instead of positive integer-order derivatives.

We now summarize below some recent investigations by Gorenflo *et al.* [36] who did indeed make references to numerous earlier *closely-related* works on this subject.

I. The Fractional (Relaxation-Oscillation) Ordinary Differential Equation

$$\frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha} + c^\alpha\, u\,(t;\alpha) = 0 \qquad (66)$$
$$(c > 0;\ 0 < \alpha \leqq 2)$$

Case I.1: Fractional Relaxation $(0 < \alpha \leqq 1)$

$$\text{Initial Condition: } u\,(0+;\alpha) = u_0$$

Case I.2: Fractional Oscillation $(1 < \alpha \leqq 2)$

$$\text{Initial Conditions: } \begin{cases} u\,(0+;\alpha) = u_0 \\ \dot{u}\,(0+;\alpha) = v_0 \end{cases}$$

with $v_0 \equiv 0$ for continuous dependence of the solution on the parameter α also in the transition from $\alpha = 1-$ to $\alpha = 1+$, $\dot{u}$ being the time-derivative of u.

Explicit Solution (in *both* cases):

$$\begin{aligned} u\,(t;\alpha) &= u_0\ E_\alpha\big(-(ct)^\alpha\big) \\ &= u_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\alpha n+1)}\,(ct)^{\alpha n} \\ &= \begin{cases} u_0\left(1 - \dfrac{(ct)^\alpha}{\Gamma(1+\alpha)}\right) \\ \approx\ u_0\ \exp\left(-\dfrac{(ct)^\alpha}{\Gamma(1+\alpha)}\right) & (t \to 0+) \\ \dfrac{u_0}{(ct)^\alpha\Gamma(1-\alpha)} & (t \to \infty), \end{cases} \end{aligned}$$

where $E_\alpha\,(z)$ denotes the familiar Mittag-Leffler function defined, as in (9), by (see, for example, [129, p. 42, Equation II.5 (23)])

$$\begin{aligned} E_\alpha\,(z) &:= \sum_{n=0}^{\infty} \frac{z^n}{\Gamma\,(\alpha n+1)} \\ &= \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{(0+)} \frac{w^{\alpha-1}\,e^w}{w^\alpha - z}\,\mathrm{d}w \\ &\quad (\alpha > 0;\ z \in \mathbb{C})\,. \end{aligned}$$

II. The Fractional (Diffusion-Wave) Partial Differential Equation

$$\frac{\partial^{2\beta} u}{\partial t^{2\beta}} = k\,\frac{\partial^2 u}{\partial x^2} \qquad (67)$$
$$(k > 0;\ -\infty < x < \infty;\ 0 < \beta \leqq 1)\,,$$

where $u = u\,(x,t;\beta)$ is assumed to be a *causal* function of time $(t > 0)$ *with*

$$u\,(\mp\infty, t;\beta) = 0.$$

Case II.1: Fractional Diffusion $\left(0 < \beta \leqq \frac{1}{2}\right)$

$$\text{Initial Condition: } u\,(x,0+;\beta) = f\,(x)$$

Case II.2: Fractional Wave $\left(\frac{1}{2} < \beta \leqq 1\right)$

$$\text{Initial Conditions: } \begin{cases} u\,(x,0+;\beta) = f\,(x) \\ \dot{u}\,(x,0+;\beta) = g\,(x) \end{cases}$$

with $g\,(x) \equiv 0$ for continuous dependence of the solution on the parameter β also in the transition from $\beta = \frac{1}{2}-$ to $\beta = \frac{1}{2}+$.

Explicit Solution (in both cases):

$$u(x,t;\beta) = \int_{-\infty}^{\infty} \mathcal{G}_c(\xi,t;\beta)\ f(x-\xi)\ d\xi, \quad (68)$$

where the Green function $\mathcal{G}_c(x,t;\beta)$ is given by

$$|x|\ \mathcal{G}_c(x,t;\beta) = \frac{z}{2}\sum_{n=0}^{\infty} \frac{(-z)^n}{n!\,\Gamma(1-\beta-\beta n)} \quad (69)$$

$$\left(z = \frac{|x|}{\sqrt{k t^{\beta}}};\ 0<\beta<1\right),$$

which can readily be expressed in terms of Wright's generalized Bessel function or the Bessel-Wright function $J_\nu^\mu(z)$ defined by (see, for example, [129, p. 42, Equation II.5 (22)])

$$J_\nu^\mu(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\,\Gamma(\mu n+\nu+1)}$$
$$= {}_0\Psi_2\left[\begin{matrix} \overline{\qquad\qquad}; \\ (1,1),(\nu+1,\mu); \end{matrix}\ -z\right]. \quad (70)$$

Example 2. Fractional-order kinetic equations of different forms have been widely used, in recent years, in the modeling and analysis of several important problems of physics and astrophysics. In fact, during the past decade or so, fractional-order kinetic equations seem to have gained popularity due mainly to the discovery of their relation with the theory of CTRW (Continuous-Time Random Walks) in [46]. These equations are investigated with the objective to determine and interpret certain physical phenomena which govern such processes as diffusion in porous media, reaction and relaxation in complex systems, anomalous diffusion, and so on (see also [43], [59] and [107]).

For an arbitrary reaction, which is characterized by a time-dependent quantity $N = N(t)$, it is possible to calculate the rate of change $\frac{dN}{dt}$ to be a balance between the destruction rate $\mathfrak{d}$ and the production rate $\mathfrak{p}$ of N, that is,

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\mathfrak{d} + \mathfrak{p}.$$

By means of feedback or other interaction mechanism, the destruction and the production depend on the quantity N itself, that is,

$$\mathfrak{d} = \mathfrak{d}(N) \qquad \text{and} \qquad \mathfrak{p} = \mathfrak{p}(N).$$

Since the destruction or the production at a time t depends not only on $N(t)$, but also on the past history $N(\eta)$ $(\eta < t)$ of the variable N, such dependence is, in general, complicated. This may be formally represented by the following equation (see [39]):

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\mathfrak{d}(N_t) + \mathfrak{p}(N_t), \quad (71)$$

where N_t denotes the function defined by

$$N_t(t^*) = N(t-t^*) \qquad (t^*>0).$$

Haubold and Mathai [39] studied a special case of the equation (71) in the following form:

$$\frac{\mathrm{d}N_j}{\mathrm{d}t} = -c_j\, N_j(t), \quad (72)$$

that is,

$$\frac{dN_j(t)}{N_j(t)} = -c_j\, \mathrm{d}t, \quad (73)$$

with the initial condition that

$$N_j(t)\big|_{t=0} = N_0,$$

is the number density of species j at time $t=0$ and the constant $c_j > 0$. This is known as a standard kinetic equation. The solution of the equation (72) is easily seen to be given by

$$N_j(t) = N_0\, \mathrm{e}^{-c_j t}, \quad (74)$$

which, upon integration, yields the following alternative form of the equation (72):

$$N(t) - N_0 = c \cdot {}_0D_t^{-1}\{N(t)\}, \quad (75)$$

where ${}_0D_t^{-1}$ is the standard integral operator and c is a constant of integration.

The fractional-order generalization of the equation (75) is given as in the following form (see [39]):

$$N(t) - N_0 = c^\nu \left({}^{\mathrm{RL}}I_{0+}^{\nu} N\right)(t) \quad (76)$$

in terms of the familiar right-sided Riemann-Liouville fractional integral operator of order ν defined, as in (54), by (see, for example, [54] and [70]; see also [26]) defined by

$$\left({}^{\mathrm{RL}}I_{0+}^{\nu} f\right)(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-u)^{\nu-1}\ f(u)\ du \quad (77)$$

$$\left(t>0;\ \Re(\nu)>0\right).$$

For a considerably large number of extensions and further generalizations of the fractional-order kinetic equation (76), the interested reader should refer to [59] and [107] as well as the other relevant references which are cited in each of these publications.

Here, in this example, we propose to investigate solution of a family of fractional-order kinetic equations which are associated with the general function $\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa)$ defined by (32), which we have introduced in this article. The results presented here are general enough and capable of being specialized appropriately to include solutions of the corresponding (known or new) fractional-order kinetic equations associated with simpler functions.

Theorem 1. *Let $c, \mu, \nu, \rho, \sigma \in \mathbb{R}^+$. Suppose also that the general function $\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa)$, defined by (32), exists. Then the solution of the following generalized fractional kinetic equation:*

$$N(t) - N_0\, t^{\mu-1}\, \mathcal{E}_{\alpha,\beta}\big(\varphi; zt^{\nu}, s, \kappa\big) = -c^{\rho}\, \big({}^{\mathrm{RL}}I^{\sigma}_{0+} N\big)(t) \tag{78}$$

is given by

$$N(t) = N_0\, t^{\mu-1} \sum_{r=0}^{\infty} (-c^{\rho} t^{\sigma})^r \cdot \sum_{k=0}^{\infty} \frac{\varphi(k)\, \Gamma(\nu k+\mu)}{(k+\kappa)^s\, \Gamma(\alpha k+\beta)\, \Gamma(\nu k+\sigma r+\mu)}\, (zt^{\nu})^k \tag{79}$$

$$(t > 0),$$

provided that the right-hand side of the solution asserted by (79) exists.

Proof. First of all, by the Laplace Convolution Theorem, it is observed from the definition (77) that

$$\begin{aligned}
&\mathcal{L}\left\{\big({}^{\mathrm{RL}}I^{\sigma}_{0+} N\big)(t) : \mathfrak{s}\right\} \\
&= \int_0^{\infty} \mathrm{e}^{-\mathfrak{s}\tau} \left(\frac{1}{\Gamma(\sigma)} \int_0^t (t-\tau)^{\sigma-1} N(\tau)\, \mathrm{d}\tau\right) \mathrm{d}t \\
&= \mathcal{L}\left\{\frac{\tau^{\sigma-1}}{\Gamma(\sigma)} : \mathfrak{s}\right\} \cdot \mathcal{L}\{N(\tau) : \mathfrak{s}\} \\
&= \frac{\mathcal{N}(\mathfrak{s})}{\mathfrak{s}^{\sigma}} \qquad \big(\Re(\mathfrak{s}) > 0;\ \Re(\sigma) > 0\big),
\end{aligned} \tag{80}$$

where

$$\mathcal{N}(\mathfrak{s}) := \mathcal{L}\{N(t) : \mathfrak{s}\} = \int_0^{\infty} \mathrm{e}^{-\mathfrak{s}t}\, N(t)\, \mathrm{d}t. \tag{81}$$

Thus, in view of the Laplace transform formula (48), we find upon taking the Laplace transform of each member of the generalized fractional kinetic equation (78) that

$$\mathcal{N}(\mathfrak{s}) - \frac{N_0}{\mathfrak{s}^{\mu}} \sum_{k=0}^{\infty} \frac{\varphi(k)\, \Gamma(\nu k+\mu)}{(k+\kappa)^s\, \Gamma(\alpha k+\beta)} \left(\frac{z}{\mathfrak{s}^{\nu}}\right)^k = -\frac{c^{\rho}}{\mathfrak{s}^{\sigma}}\, \mathcal{N}(\mathfrak{s})$$

$\big(\Re(\mathfrak{s}) > 0;\ \Re(\mu) > 0;\ \Re(\nu) > 0;\ \Re(\sigma) > 0;\ \Re(\alpha) > 0\big)$, so that

$$\left(1 + \frac{c^{\rho}}{\mathfrak{s}^{\sigma}}\right) \mathcal{N}(\mathfrak{s}) = N_0 \sum_{k=0}^{\infty} \frac{\varphi(k)\, \Gamma(\nu k+\mu)}{(k+\kappa)^s\, \Gamma(\alpha k+\beta)}\, \frac{z^k}{\mathfrak{s}^{\nu k+\mu}},$$

that is, that

$$\begin{aligned}
\mathcal{N}(\mathfrak{s}) &= N_0 \left(1 + \frac{c^{\rho}}{\mathfrak{s}^{\sigma}}\right)^{-1} \cdot \sum_{k=0}^{\infty} \frac{\varphi(k)\, \Gamma(\nu k+\mu)}{(k+\kappa)^s\, \Gamma(\alpha k+\beta)}\, \frac{z^k}{\mathfrak{s}^{\nu k+\mu}} \\
&= N_0 \sum_{r=0}^{\infty} (-1)^r \left(\frac{c^{\rho}}{\mathfrak{s}^{\sigma}}\right)^r \cdot \sum_{k=0}^{\infty} \frac{\varphi(k)\, \Gamma(\nu k+\mu)}{(k+\kappa)^s\, \Gamma(\alpha k+\beta)}\, \frac{z^k}{\mathfrak{s}^{\nu k+\mu}} \\
&= N_0 \sum_{r=0}^{\infty} (-c^{\rho})^r \cdot \sum_{k=0}^{\infty} \frac{\varphi(k)\, \Gamma(\nu k+\mu)}{(k+\kappa)^s\, \Gamma(\alpha k+\beta)}\, \frac{z^k}{\mathfrak{s}^{\sigma r+\nu k+\mu}}
\end{aligned} \tag{82}$$

$\big(\Re(\mathfrak{s}) > 0;\ \Re(\mu) > 0;\ \Re(\nu) > 0;\ \Re(\sigma) > 0;\ \Re(\alpha) > 0\big)$, where we have used the following geometric series:

$$\left(1 + \frac{c^{\rho}}{\mathfrak{s}^{\sigma}}\right)^{-1} = \sum_{r=0}^{\infty} (-1)^r \left(\frac{c^{\rho}}{\mathfrak{s}^{\sigma}}\right)^r, \quad \left(\left|\frac{c^{\rho}}{\mathfrak{s}^{\sigma}}\right| < 1\right).$$

We now invert the Laplace transform occurring in (82) by using the following well-known identity:

$$\mathcal{L}\left\{t^{\lambda}:\mathfrak{s}\right\}=\frac{\Gamma(\lambda+1)}{\mathfrak{s}^{\lambda+1}} \iff \mathcal{L}^{-1}\left(\frac{1}{\mathfrak{s}^{\lambda+1}}\right)=\frac{t^{\lambda}}{\Gamma(\lambda+1)} \quad (83)$$
$$\big(\Re(\lambda)>-1;\ \Re(\mathfrak{s})>0\big).$$

We are thus led to the solution (79) asserted by Theorem 1. This evidently completes the proof of Theorem 1. □

Remark 8. The distinct advantage of using the general function $\mathcal{E}_{\alpha,\beta}(\varphi;z,s,\kappa)$, defined by (32), in the non-homogeneous term of the fractional-order kinetic equation (78) lies in its generality so that solutions of other kinetic equations involving relatively simpler non-homogeneous terms can be derived by appropriately specializing the solution (79) asserted by Theorem 1.

Theorem 2. *Let $c,\mu,\nu,\rho,\sigma\in\mathbb{R}^{+}$. Suppose also that the general function $\mathfrak{E}_{\alpha,\beta}(\phi;z)$, defined by (14), exists. Then the solution of the following generalized fractional kinetic equation:*

$$N(t)-N_0\,t^{\mu-1}\,\mathfrak{E}_{\alpha,\beta}\left(\phi;zt^{\nu}\right) = -c^{\rho}\left({}^{\mathrm{RL}}I_{0+}^{\sigma}N\right)(t) \quad (84)$$

is given by

$$N(t)=N_0\,t^{\mu-1}\sum_{r=0}^{\infty}\left(-c^{\rho}t^{\sigma}\right)^{r} \cdot\sum_{k=0}^{\infty}\frac{\phi(k)\,\Gamma(\nu k+\mu)}{\Gamma(\alpha k+\beta)\,\Gamma(\nu k+\sigma r+\mu)}\left(zt^{\nu}\right)^{k} \quad (85)$$
$$(t>0),$$

provided that the right-hand side of the solution asserted by (85) exists.

Proof. Our demonstration of Theorem 2 would run parallel that of Theorem 1. Use is made, in this case, of the definition (14) and the Laplace transform formula (50). The details are being omitted here. □

Theorem 3. *For $c,\mu,\nu,\rho,\sigma\in\mathbb{R}^{+}$, let the extended Hurwitz-Lerch zeta function:*

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}(z,s,\kappa),$$

defined by (28), exist. Then the solution of the following generalized fractional kinetic equation:

$$N(t)-N_0\,t^{\mu-1}\,\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}\left(zt^{\nu},s,\kappa\right) = -c^{\rho}\left({}^{\mathrm{RL}}I_{0+}^{\sigma}N\right)(t) \quad (86)$$

is given by

$$N(t)=N_0\,t^{\mu-1}\sum_{r=0}^{\infty}\left(-c^{\rho}t^{\sigma}\right)^{r}\frac{\Gamma(\mu)}{\Gamma(\sigma r+\mu)} \cdot\Phi_{\mu,\lambda_1,\cdots,\lambda_p;\sigma r+\mu,\mu_1,\cdots,\mu_q}^{(\nu,\rho_1,\cdots,\rho_p;\nu,\sigma_1,\cdots,\sigma_q)}\left(zt^{\nu},s,\kappa\right) \quad (87)$$
$$(t>0),$$

provided that the right-hand side of the solution asserted by (87) exists.

Proof. Theorem 3 can be proven, along the lines analogous to those of our demonstration of Theorem 1 and Theorem 3, by applying the definition (28) and the Laplace transform formula (53). We choose to skip the details invoved. □

Example 3. In this third example, we choose to recall an earlier investigation of an initial-value problem in which Hilfer (see [43]) considered the following eigenvalue equation for the general (Hilfer's) *two-parameter* fractional derivative operator ${}^{\mathrm{H}}D_{0+}^{\alpha,\beta}$ of order α $(0<\alpha<1)$ and type β $(0\leqq\beta\leqq1)$ defined by the equation (5):

$$\left({}^{\mathrm{H}}D_{0+}^{\alpha,\beta}f\right)(x)=\lambda f(x) \qquad (x>0) \quad (88)$$

under the initial condition given, in terms of the corresponding *two-parameter* fractional integral operator ${}^{\mathrm{H}}I_{0+}^{\alpha,\beta}$, by

$$\left({}^{\mathrm{H}}I_{0+}^{(1-\beta)(1-\alpha)}f\right)(0+)=c_0, \quad (89)$$

where it is tacitly assumed that

$$\left({}^{\mathrm{H}}I_{0+}^{(1-\beta)(1-\alpha)}f\right)(0+) := \lim_{x\to0+}\left\{\left({}^{\mathrm{H}}I_{0+}^{(1-\beta)(1-\alpha)}f\right)(x)\right\},$$

c_0 being a given constant and with the parameter λ being the eigenvalue. The condition $x>0$ in Eq. (88) was not mentioned explicitly by Hilfer [43, p. 115, Eq. (118)]. However, since the Riemann-Liouville, the Liouville-Caputo and the Hilfer operators of fractional calculus are all defined by definite integrals over the obviously non-empty interval $(0, x)$, such a condition as $x > 0$ is tacitly assumed to be satisfied in all developments involving each of these operators of fractional calculus.

In terms of the two-parameter Mittag-Leffler function defined by (9), Hilfer's over two decades old solution of (88) under the initial condition (89) is given by (see [43, p. 115, Eq. (124)]):

$$f(x) = c_0\, x^{(1-\beta)(\alpha-1)}\, E_{\alpha,\alpha+\beta(1-\alpha)}(\lambda x^{\alpha}). \tag{90}$$

Now, upon setting $\beta = 0$ and $c_0 = 1$ in Hilfer's solution (89) (with, of course, $x > 0$), we are led to corrected version of the claimed solution (see [148, p. 802, Eq. (3.1)]) of the following initial-value problem:

$$\left({}^{\mathrm{RL}}D_{0+}^{\alpha} f\right)(x) = \lambda f(x) \qquad (x > 0), \tag{91}$$

together with the initial condition:

$$\left({}^{\mathrm{RL}}I_{0+}^{1-\alpha} f\right)(0+) = 1, \tag{92}$$

where, just as in Eq. (89),

$$\left({}^{\mathrm{RL}}I_{0+}^{1-\alpha} f\right)(0+) := \lim_{x\to 0+} \left\{\left({}^{\mathrm{RL}}I_{0+}^{1-\alpha} f\right)(x)\right\}$$

in the form given by

$$f(x) = x^{\alpha-1}\, E_{\alpha,\alpha}(\lambda x^{\alpha}), \tag{93}$$

in terms of the two-parameter Mittag-Leffler function defined by (9).

Remark 9. In each of the above examples, we have made use of the classical Laplace transform in solving the considered fractional-order ordinary and partial differential equations. Other known or classical integral transforms can possibly also be suitably applied in some of these cases. Nevertheless, it may be immensely and potentially helpful to investigate the possibility of developing some kind of an integral or other transformation which would enable us to find solutions of fractional-order differential equations by first reducing them to the corresponding integer-order differential equations.

5. Developments in Recent Years

In recent years, a remarkably wide variety of real-world problems and issues in many areas have been modeled and analysed by making use of some very powerful tools, one of which involves applications of operators of fractional calculus. In fact, such important definitions have been introduced for fractional-order derivatives, including, for example, the Riemann-Liouville, the Grünwald-Letnikov, the Liouville-Caputo, the Caputo-Fabrizio and the Atangana-Baleanu fractional-order derivatives (see, for example, [13], [24], [26], [54], [82] and [155]).

By using the fundamental relations of the Riemann-Liouville fractional integral, the Riemann-Liouville fractional derivative was constructed, which involves the convolution of a given function and a power-law kernel (see, for details, [54] and [82]). The Liouville-Caputo (LC) fractional derivative involves the convolution of the local derivative of a given function with a power-law function [25]. Caputo and Fabrizio [24] and Atangana and Baleanu [13] proposed some interesting fractional-order derivatives based upon the exponential decay law which is a generalized power-law function (see [5], [8], [10], [11], [12] and [15]). The Caputo-Fabrizio (CFC) fractional-order derivative as well as the Atangana-Baleanu (ABC) fractional-order derivative allow us to describe complex physical problems that follow, at the same time, the power law and the exponential decay law (see, for details, [5], [8], [10], [11], [12] and [15]).

In a noteworthy earlier investigation, Srivastava and Saad [137] investigated the model of the gas dynamics equation (GDE) by extending it to some new models involving the time-fractional gas dynamics equation (TFGDE) with the Liouville-Caputo (LC), Caputo-Fabrizio (CFC) and Atangana-Baleanu (ABC) time-fractional derivatives. They employed the Homotopy Analysis Transform Method (HATM) in order to calculate the approximate solutions of TFGDE by using LC, CFC and ABC in the Liouville-Caputo sense and studied the convergence analysis of HATM by finding the interval of convergence through the h-curves. Srivastava and Saad [137] also showed the effectiveness and accuracy of this method (HATM) by comparing the approximate solutions based upon the LC, CFC and ABC time-fractional derivatives.

Consider the following *homogeneous* time-fractional gas dynamics equation (TFGDE):

$$\frac{\partial^\alpha \psi}{\partial \tau^\alpha} + \psi \, \frac{\partial \psi}{\partial \varsigma} - \psi(1-\psi) = 0, \tag{94}$$

where $(\varsigma, \tau) \in (0, \infty) \times (0, \tau_0)$ and $0 < \alpha \leqq 1$.

Srivastava and Saad [137] used the HATM (see, for example, [58] and [90]) in order to solve the LC, CFC and ABC analogues of the TFGDE (94). They obtained these analogous equations by replacing the time-fractional derivative $\dfrac{\partial^\alpha \psi}{\partial \tau^\alpha}$ in the TFGDE (94) by

$${}^{\text{LC}}_{\ 0}D^\alpha_\tau \psi, \qquad {}^{\text{CFC}}_{\ \ 0}D^\alpha_\tau \psi \qquad \text{and} \qquad {}^{\text{ABC}}_{\ \ 0}D^\alpha_\tau \psi,$$

successively, where the order α of the time-fractional derivatives is constrained by

$$n - 1 < \alpha \leqq n \qquad (n \in \mathbb{N}).$$

The corresponding LC, CFC and ABC time-fractional analogues of the TFGDE (94) are given by

$${}^{\text{LC}}_{\ 0}D^\alpha_\tau \psi + \psi \, \frac{\partial \psi}{\partial \varsigma} - \psi(1-\psi) = 0 \tag{95}$$
$$(0 < \alpha \leqq 1;\ \varsigma \in \mathbb{R};\ \tau > 0),$$

$${}^{\text{CFC}}_{\ \ 0}D^\alpha_\tau \psi + \psi \, \frac{\partial \psi}{\partial \varsigma} - \psi(1-\psi) = 0 \tag{96}$$
$$(0 < \alpha \leqq 1;\ \varsigma \in \mathbb{R};\ \tau > 0)$$

and

$${}^{\text{ABC}}_{\ \ 0}D^\alpha_\tau \psi + \psi \, \frac{\partial \psi}{\partial \varsigma} - \psi(1-\psi) = 0 \tag{97}$$
$$(0 < \alpha \leqq 1;\ \varsigma \in \mathbb{R};\ \tau > 0),$$

respectively. Here

$${}^{\text{LC}}_{\ 0}D^\alpha_\tau \qquad \text{and} \qquad {}^{\text{CFC}}_{\ \ 0}D^\alpha_\tau$$

denote the time-fractional derivatives of order α for a suitably defined function $\mathfrak{f}(\tau)$, which are defined, respectively, by

$$\begin{aligned} {}^{\text{LC}}_{\ 0}D^\alpha_\tau\big(\mathfrak{f}(\tau)\big) &= J^{m-\alpha} D^m\big(\mathfrak{f}(\tau)\big) \\ &= \frac{1}{\Gamma(m-\alpha)} \int_0^\tau (\tau - t)^{m-\alpha-1}\, \mathfrak{f}^{(m)}(t)\, \mathrm{d}t \end{aligned}$$
$$(m - 1 < \alpha \leqq m;\ m \in \mathbb{N};\ \mathfrak{f} \in C^m_\mu;\ \mu \geqq -1)$$

and

$$\begin{aligned} &{}^{\text{CFC}}_{\ \ 0}D^\alpha_\tau\big(\mathfrak{f}(\tau)\big) \\ &\quad = \frac{M(\alpha)}{1-\alpha} \int_0^\tau \exp\left(-\frac{\alpha(\tau - t)}{1-\alpha}\right) D\big(\mathfrak{f}(t)\big)\, \mathrm{d}t \end{aligned}$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$ and ${}^{\text{ABC}}_{\ \ 0}D^\alpha_\tau\big(\mathfrak{f}(\tau)\big)$ is known as the ABC time-fractional derivative of order α in the Liouville-Caputo sense given, for a suitably defined function $\mathfrak{f}(\tau)$, by

$$\begin{aligned} &{}^{\text{ABC}}_{\ \ 0}D^\alpha_\tau\big(\mathfrak{f}(\tau)\big) \\ &\quad = \frac{M(\alpha)}{1-\alpha} \int_0^\tau E_\alpha\left(-\frac{\alpha(\tau - t)}{1-\alpha}\right) D\big(\mathfrak{f}(t)\big)\, \mathrm{d}t, \end{aligned}$$

where

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$$

is the Mittag-Leffler function and $M(\alpha)$ is a normalization function with the same properties as in the Liouville-Caputo (LC) and the Caputo-Fabrizio (CFC) cases. For the details of this and other closely-related investigations, the interested reader should see the work by Srivastava and Saad [137]).

In the current onslaught of the Corona virus, which is referred to as COVID-19 (see, for details, [2], [76], [83] and [99]). As in the case of the Corona virus, the Ebola virus can be transmitted to others by contact with infected body fluids, through broken skin, or through the mucous membranes of the eyes, nose and mouth, but the Ebola virus can also be transmitted through sexual contact with a person who has the virus or has recovered from it (see, for details, [20]; see also the recently-published works [75], [81], [108], [121], [123], [124], [127], and [145] for the fractional-order modeling of other diseases and other biological situations).

Fractional calculus is a generalization of the classical (or ordinary) calculus and many researchers have paid attention to this science as and when they encounter a number of issues in the real world. Most of these issues do not have exact analytical solution. This situation naturally interests many researchers to look for and apply numerical and approximate methods to obtain solutions by using such methods. There are many useful methods, such as the homotopy analysis (see [29], [30] [89] and [63]), He's variational iteration method (see [41] and [43]), Adomian's decomposition method (see [45], [44] and [46]), the Fourier spectral methods [47], finite difference schemes (see [48]), collocation methods (see [52], [53] and [88]), and so on. In order to find more about the fractal calculus, we refer the readers to the investigations in [54] and [82]. More recently, a new concept was introduced for the fractional-order operator because this operator has two orders, the first representing the fractional order and the second representing the fractal dimension. Some recent developments in the area of numerical techniques can be found in (for example) [91], [140] and [141].

Our attention in this section is now drawn toward the idea of the fractal-fractional derivative of the composite order (ρ, k) on the fractal-fractional Ebola virus (FFEV). With this object in view (see [9]), we replace the derivative of integer order with respect to ζ by the fractal-fractional derivatives based on the power law (FFP), the exponential-law (FFE) and the Mittag-Leffler law (FFM) kernels which correspond to the Liouville-Caputo (LC) (see [84]), Caputo-Fabrizio (CF) (see [93]) and the Atangana-Baleanu (AB) (see [94]) fractional derivatives, respectively. Here, just as we have already mentioned in Section 4 above, we use the term *Liouvile-Caputo fractional derivative* in order to give due credit also to Liouville who, in fact, considered such fractional derivatives many decades earlier in 1832. This topic has attracted many researchers and has been applied to researches stemming from various real-world situations (see, for example, [14], [34], [62] and [149]).

During the past several years, many researchers' focus has been directed towards modeling and analysis of various problems in biomathematical sciences. This branch of science represents many distinguished data on biological phenomena such as the Ebola and other related viruses, the nervous system and its impulse transmission, the bacterial cell and its spread, *et cetera* (see [41] and [138]). This has led to the modeling of many real-world problems. As a result of problems that arise from the real world on the basis of statistical analysis and biological experiments, mathematical models of these problems are proposed and most of them were studied. These proposed models enable scientists and researchers to study and verify the behavior of these models separately in a biological laboratory experiment (see [7], [21], [57] and [85]). After modeling the biological phenomenon mathematically, that is, as a function of time and the parameters involved, the numerical solutions can be found and these solutions can then be represented in tables and figures. Also, if the laboratory results are available, comparison between theoretical and laboratory results can be made. The parameters affecting this system can also be controlled appropriately. Also, one of the advantages of mathematical modeling is the possibility of re-studying the problems many times and at any time value without re-experimenting.

We begin by introducing the epidemiological model of the Ebola virus as follows:

$$D_\zeta \beta_1(\zeta) = -\alpha\, \beta_1(\zeta)\, \beta_2(\zeta) + \beta\, \beta_3(\zeta) - \gamma\, N, \tag{98}$$

$$D_\zeta \beta_2(\zeta) = \alpha\, \beta_1(\zeta)\, \beta_2(\zeta) - \epsilon \beta_2(\zeta) - \delta \beta_2(\zeta), \tag{99}$$

$$D_\zeta \beta_3(\beta) = \delta \beta_2(\zeta) - \beta \beta_3(\zeta) \tag{100}$$

and

$$D_\zeta \beta_4(\zeta) = \epsilon \beta_2(\zeta) + \gamma N, \tag{101}$$

where, as usual, $D_\zeta = \frac{\mathrm{d}}{\mathrm{d}\zeta}$ and $\zeta \geqq 0$.

In the following table, we define the independent variables and the parameters of the Ebola virus.

Symbol	definition
$\beta_1(\zeta)$	The susceptible population
$\beta_2(\zeta)$	The infected population
$\beta_3(\zeta)$	The recovery population
$\beta_4(\zeta)$	The population died in the region
N	The total population in the region
α	The rate of infection with the disease
β	The rate of susceptibility
γ	The rate of natural death
ϵ	The rate of death from the disease
δ	The rate of recovery from the disease

The new model is obtained upon replacing the ordinary derivative D_ζ in the above epidemiological model in (98) to (101) by the corresponding fractal-fractional derivative involving the power law kernel as in the earlier work (see [9]).

$$ {}^{\mathrm{FFP}}_{\ 0}D^{\rho,k}_{\zeta}\beta_1(\zeta) = -\alpha\,\beta_1(\zeta)\,\beta_2(\zeta) + \beta\,\beta_3(\zeta) - \gamma\,N, \tag{102} $$

$$ {}^{\mathrm{FFP}}_{\ 0}D^{\rho,k}_{\zeta}\beta_2(\zeta) = \alpha\,\beta_1(\zeta)\,\beta_2(\zeta) - \epsilon\beta_2(\zeta) - \delta\beta_2(\zeta), \tag{103} $$

$$ {}^{\mathrm{FFP}}_{\ 0}D^{\rho,k}_{\zeta}\beta_3(\zeta) = \delta\beta_2(\zeta) - \beta\beta_3(\zeta) \tag{104} $$

and

$$ {}^{\mathrm{FFP}}_{\ 0}D^{\rho,k}_{\zeta}\beta_4(\zeta) = \epsilon\beta_2(\zeta) + \gamma N \qquad \big(\rho, k \in (0,1]\big), \tag{105} $$

where the functions $\beta_i(\zeta)$ $(i = 1,2,3,4)$ are continuous in the interval (a,b) and fractal differentiable on (a,b) with order k. The fractal-fractional derivative of $\beta_i(\zeta)$ of order ρ in the Liouville-Caputo (LC) sense with the power law are given by (see [9])

$$ {}^{\mathrm{FFP}}_{\ 0}D^{\rho,k}_{\zeta}\beta_i(\zeta) = \frac{1}{\Gamma(1-\rho)}\frac{\mathrm{d}}{\mathrm{d}\zeta^k}\int_0^{\zeta}(\zeta-\tau)^{-\rho}\beta_i(\tau)\mathrm{d}\tau \tag{106} $$

$$ (0 < \rho, k \leqq 1;\ i = 1,2,3,4) $$

and

$$ \frac{d\beta_i(\zeta)}{d\zeta^k} = \lim_{\tau\to\zeta}\left\{\frac{\beta_i(\tau)-\beta_i(\zeta)}{\tau^k-\zeta^k}\right\}. \tag{107} $$

Just as we pointed out above (and also in Section 4), we have used the term “Liouvile-Caputo sense" in order to give due credit also to Liouville who considered such fractional derivatives many decades earlier in 1832.

For the relevant details of the numerical solutions for the above model of the fractal-fractional Ebola virus, the reader is referred to the work of Srivastava and Saad [139]. The detailed analysis of this model and the numerical solutions, which are presented in [139] and in other works cited in [139] are potentially beneficial to biological researchers with a view to linking these findings to the biological laboratory results.

Finally, in this section, we turn to the fact that many experiments and theories have shown that a large number of abnormal phenomena that occurs in the engineering and applied sciences can be well-described by using discrete fractional calculus. In particular, fractional difference equations have been found to provide powerful tools in the modeling and analysis of various phenomena in many different fields of science and engineering such as those in, for example, physics, fluid mechanics and heat conduction. Considerable attention has been given in the existing literature to the subject of fractional difference equations on the finite time scales (see, for example, [35]). In the current literature on this subject, there are a few papers which investigate the existence and uniqueness of fractional difference equations in the sense of the Riemann-Liouville (RL) fractional calculus.

We choose to recall here the work of Lu *et al.* [65] investigated the existence and uniqueness of the following uncertain fractional forward difference equation (UFFDE) given by

$$ \left({}^{\mathrm{RL}}_{\psi-1}\Delta^{\psi}\wp\right)(z) = \mathcal{H}_1\big(z+\psi, \wp(z+\psi)\big) + \mathcal{H}_2\big(z+\psi, \wp(z+\psi)\big)\varepsilon_{z+\psi} \tag{108} $$

and

$$\left({}^{\mathrm{RL}}_{\psi-1}\Delta^{-(1-\psi)}\wp\right)(z)\Big|_{z=0} =: a_0, \tag{109}$$

where ${}^{\mathrm{RL}}_{\psi-1}\Delta^{\psi}$ denotes fractional Riemann-Liouville type forward difference with $0 < \psi \leqq 1$, and $\mathcal{H}_1$ and $\mathcal{H}_2$ are two real-valued functions defined on $[1, \infty] \times \mathcal{R}$, $z \in \mathcal{N}_0 \cap [0, \mathcal{T}]$, $a_0 \in \mathcal{R}$ is a crisp number, and $\varepsilon_\psi, \varepsilon_{\psi+1}, \cdots, \varepsilon_{\mathcal{T}+\psi}$ are $(\mathcal{T}+1)$ IID uncertain variables with symmetrical uncertainty distribution $\mathbf{L}(\ell_1, \ell_2)$. The above work was generalized by Mohammed [72] and, subsequently, by Srivastava and Mohammed [133]. In addition, Mohammed *et al.* [73] obtained the existence and uniqueness of the nabla case (backward) of the equations (108) and (109). Motivated by these developments, Srivastava *et al.* [134] considered a general family of uncertain fractional difference equations of the Liouville-Caputo type (UFLCDE). They derived an uncertain fractional sum equation, which is equivalent to the UFLCDE by using the basic properties of the Liouville-Caputo type uncertain fractional difference equations. After introducing a successive Picard iteration method for finding a solution to the UFLCDE, Srivastava *et al.* [134] applied the theory of Banach contraction under the Lipschitz constant condition and successfully investigated the structure of the algebras of existence and uniqueness of the UFLCDE. They also presented three examples to show the effectiveness of the proposed investigation (see, for details, Srivastava *et al.* [134]).

6. Concluding Remarks and Observations

The main object of this survey-cum-expository review article is focussed toward the widespread applications and usages of many of the currently-investigated operators of fractional calculus (that is, fractional-order integrals and fractional-order derivatives) in the modeling and analysis of a remarkably wide variety of applied scientific and real-world problems in mathematical, physical, biological, engineering and statistical sciences as well as in other scientific disciplines. Here, in this review article, we have presented a brief introductory overview of the theory and applications of the fractional-calculus operators which are based upon the general Fox-Wright function, which we have presented together with the related historical background, and its such specialized forms as the Mittag-Leffler type functions.

In the bibliography of this presentation, we have chosen to include a considerably large number of recently-published books, monographs and edited volumes (as well as journal articles) dealing with the extensively-investigated subject of fractional calculus and its widespread applications. Indeed, judging by the on-going contributions to the theory and applications of *Fractional Calculus and Its Applications*, which are continuing to appear in some of the leading journals of mathematical, physical, statistical and engineering sciences, the importance of the subject-matter dealt with in this survey-cum-expository review article cannot be over-emphasized. Moreover, for the potential use of those of the readers who are interested in pursuing investigations on the subject of fractional calculus, we give here references to some of the other applications of the operators of fractional calculus in the mathematical sciences, which are not mentioned in the preceding sections.

(i) Theory of Generating Functions of Orthogonal Polynomials and Special Functions (see, for details, [132]);

(ii) Geometric Function Theory of Complex Analysis (especially the Theory of Analytic, Univalent, and Multivalent Functions) (see, for details, [135] and [136]);

(iii) Integral Equations (see, for details, [37], [116] and [117]);

(iv) Integral Transforms (see, for details, [55] and [69]);

(v) Generalized Functions (see, for details, [69]);

(vi) Theory of Potentials (see, for details, [87]).

There is a fastly-growing trend now-a-days to investigate and apply the fractional-order quantum or basic (or $\mathfrak{q}$-) calculus not only in the aforementioned Geometric Function Theory of Complex Analysis (see, for detailed historical and introductory overview, the recently-published survey-cum-expository review article [109]), but indeed also in medeling and analysis of applied problems as well as in extending the well-established theory and applications of various rather classical mathematical inequalities. Unfortunately, however, some of mostly amateurish-type reseachers on these and other related topics continue to produce obvious and inconsequential variations of the known $\mathfrak{q}$-results in terms of the so-called $(\mathfrak{p}, \mathfrak{q})$-calculus by forcing a redundant or superfluous parameter $\mathfrak{p}$ into the classical $\mathfrak{q}$-calculus and thereby falsely claiming "generalization" (see [109, p. 340] and [110, pp. 1511–1512]).

With a view to motivating further researches dealing with various widespread applications of fractional-order modelling and analysis, we find it to be worthwhile to cite several recently-published works (see, for example, [50], [111], [112], [131] and [146]; see also the references to the related earlier developments which are cited in each of these works).

We choose to conclude this presentation by reiterating the fact that the extensively-investigated *and* celebrated special function named after the famous Swedish mathematician, Magnus Gustaf (Gösta) Mittag-Leffter (16 March 1846–07 July 1927), as well as its various extensions and generalizations including (among others) those that are considered here, have found remarkable applications in the solutions of a significantly wide variety of problems in the physical, biological, chemical, earth and engineering sciences (see, for example, [43] and [54]). However, in a presentation of this *modest* size, it is naturally hard to justify and elaborate upon the tremendous potential for applications of all those Mittag-Leffler type functions in one and more variables which have appeared in the existing literature on the subject. In our presentation here, we have focussed mainly on the problems and prospects involving some of the Mittag-Leffler type functions in the areas of various families of fractional differential and integro-differential equations.

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References

[1] Abramowitz, M., Stegun I. A., (Editors). (1965). Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, *Tenth Printing, National Bureau of Standards, Applied Mathematics Series, 55*, National Bureau of Standards, Washington, D.C., (1972); Reprinted by Dover Publications, New York, (1965) (see also [78]).

[2] Abdo, M. S., Shah, K., Wahash, H. A., & Panchal, S. K. (2020). On a comprehensive model of the novel coronavirus (COVID-19) under Mittag-Leffler derivative. *Chaos, Solitons & Fractals, 135*, 109867.

[3] Ahmad, B., Henderson, J., & Luca, R. (2021). Boundary Value Problems for Fractional Differential Equations and Systems, *Trends in Abstract and Applied Analysis, 9,* World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong.

[4] Ahmadova, A., Huseynov, I. T., Fernandez, A., & Mahmudov, N. I. (2021). Trivariate Mittag-Leffler functions used to solve multi-order systems of fractional differential equations. *Communications in Nonlinear Science and Numerical Simulation, 97*, 105735.

[5] Alsaedi, A., Baleanu, D., Etemad, S., & Rezapour, S. (2016). On coupled systems of time-fractional differential problems by using a new fractional derivative. *Journal of Function Spaces, 2016*, 4626940.

[6] Anastassiou, G. A. (2021). Generalized Fractional Calculus: New Advancements and Applications, *Studies in Systems, Decision and Control, 305*, Springer, Cham.

[7] Area, I., Batarfi, H., Losada, J., Nieto, J. J., Shammakh, W., & Torres, Á. (2015). On a fractional order Ebola epidemic model. *Advances in Difference Equations, 2015*(1), 1–12.

[8] Atangana, A. (2016). On the new fractional derivative and application to nonlinear Fisher's reaction–diffusion equation. *Applied Mathematics and Computation, 273*, 948–956.

[9] Atangana, A. (2017). Fractal-fractional differentiation and integration: Connecting fractal calculus and fractional calculus to predict complex system. *Chaos, solitons & fractals, 102*, 396–406.

[10] Atangana, A., & Alkahtani, B. S. T. (2015). Extension of the resistance, inductance, capacitance electrical circuit to fractional derivative without singular kernel. *Advances in Mechanical Engineering, 7*(6), 1687814015591937.

[11] Atangana, A., & Alkahtani, B. S. T. (2015). Analysis of the Keller–Segel model with a fractional derivative without singular kernel. *Entropy, 17*(6), 4439–4453.

[12] Atangana, A., & Alkahtani, B. S. T. (2016). New model of groundwater flowing within a confine aquifer: Application of Caputo-Fabrizio derivative. *Arabian Journal of Geosciences, 9*(1), 1–6.

[13] Atangana, A., & Baleanu, D. (2016). New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model, *Thermal Science, 20*, 763–769.

[14] Atangana, A., & Khan, M. A. (2020). Modeling and analysis of competition model of bank data with fractal-fractional Caputo-Fabrizio operator. *Alexandria Engineering Journal, 59*(4), 1985–1998.

[15] Atangana, A., & Nieto, J. J. (2015). Numerical solution for the model of RLC circuit via the fractional derivative without singular kernel. *Advances in Mechanical Engineering, 7*(10), 1687814015613758.

[16] Badík A., Fečkan, M. (2021). Applying fractional calculus to analyze final consumption and gross investment influence on GDP, *Journal of Applied Mathematics, Statistics and Informatics, 17*(1), 65–72.

[17] Baleanu, D., Diethelm, K., Scalas, E., & Trujillo, J. J. (2012). Fractional Calculus: Models and Numerical Methods (*Vol. 3*). *World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong.*

[18] Baleanu, D., & Mustafa, O. G. (2015). Asymptotic Integration and Stability: For Ordinary, Functional and Discrete Differential Equations of Fractional Order (*Vol. 4*). *World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong.*

[19] Barnes, E. W. (1906). V. The asymptotic expansion of integral functions defined by Taylor's series. *Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character, 206*(402-412), 249–297.

[20] Baseler, L., Chertow, D. S., Johnson, K. M., Feldmann, H., & Morens, D. M. (2017). The pathogenesis of Ebola virus disease. *Annual Review of Pathology: Mechanisms of Disease, 12*, 387–418.

[21] Bonyah, E., Badu, K., & Asiedu-Addo, S. K. (2016). Optimal control application to an Ebola model. *Asian Pacific Journal of Tropical Biomedicine, 6*(4), 283–289.

[22] Buschman, R. G., & Srivastava, H. M. (1990). The $\overline{H}$ function associated with a certain class of Feynman integrals. *Journal of Physics A: Mathematical and General, 23*(20), 4707–4710.

[23] Caputo, M. (1969). Elasticita de dissipazione, Zanichelli, Bologna, Italy, (Links). *SIAM Journal on Numerical Analysis.*

[24] Caputo, M., & Fabrizio, M. (2015). A new definition of fractional derivative without singular kernel. *Progress in Fractional Differentiation and Applications, 1*(2), 1–13.

[25] Caputo, M., & Mainardi, F. (1971). A new dissipation model based on memory mechanism. *Pure and Applied Geophysics, 91*(1), 134–147.

[26] Cattani, C., Srivastava, H. M., Yang, X.-J. (Editors). (2015). Fractional Dynamics. *Emerging Science Publishers (De Gruyter Open), Berlin and Warsaw.*

[27] Chen, S.-B., Jahanshahi, H., Abba, O. A., Solís-Pérez, J. E., Bekiros, S., Gómez-Aguilar, J. F., Yousefpour, A., & Chu, Y.-M. (2020). The effect of market confidence on a financial system from the perspective of fractional calculus: Numerical investigation and circuit realization, *Chaos, Solitons & Fractals, 140*, 110223.

[28] Diethelm, K. (2010). The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type. *Springer Science & Business Media, Berlin, Heidelberg and New York.*

[29] Erdélyi, A., Magnus, W., Oberhettinger, F., & Tricomi, F. G. (1954). Tables of Integral Transforms, *Vol. I, McGraw-Hill Book Company, New York, Toronto and London.*

[30] Erdélyi, A., Magnus, W., Oberhettinger, F., & Tricomi, F. G. (1954). Tables of Integral Transforms, *Vol. II, McGraw-Hill Book Company, New York, Toronto and London.*

[31] Fahad, H. M., & Fernandez, A. (2021). Operational calculus for Caputo fractional calculus with respect to functions and the associated fractional differential equations. *Applied Mathematics and Computation, 409*, 126400.

[32] Fernandez, A., Baleanu, D., & Srivastava, H. M. (2019). Series representations for fractional-calculus operators involving generalised Mittag-Leffler functions. *Communications in Nonlinear Science and Numerical Simulation, 67*, 517–527; see also Corrigendum, *Communications in Nonlinear Science and Numerical Simulation, 82*, 1–1, (2020).

[33] Garrappa, R., Giusti, A., & Mainardi, F. (2021). Variable-order fractional calculus: A change of perspective. *Communications in Nonlinear Science and Numerical Simulation, 102*, 105904.

[34] Gómez-Aguilar, J. F., & Atangana, A. (2021). New chaotic attractors: Application of fractal-fractional differentiation and integration. *Mathematical Methods in the Applied Sciences, 44*(4), 3036–3065.

[35] Goodrich, C., & Peterson, A. C. (2015). Discrete Fractional Calculus (*Vol. 1350*). *Springer, Berlin, Heidelberg and New York.*

[36] Gorenflo, R., Mainardi, F., & Srivastava, H. M. (2020). Special functions in fractional relaxation-oscillation and fractional diffusion-wave phenomena. *In Proceedings of the Eighth International Colloquium on Differential Equations, Plovdiv, Bulgaria, 18–23 August, 1997* (pp. 195–202). De Gruyter, Berlin, Basel and Boston.

[37] Gorenflo, R., & Vessella, S. (1991). Abel Integral Equations. (*Vol. 1461, pp. viii+-215*). *Springer, Berlin, Heidelberg and New York.*

[38] Guariglia, E. (2021). Fractional calculus, zeta functions and Shannon entropy. *Open Mathematics, 19*(1), 87–100.

[39] Haubold, H. J., & Mathai, A. M. (2000). The fractional kinetic equation and thermonuclear functions. *Astrophysics and Space Science, 273*(1), 53–63.

[40] Haubold, H. J., Mathai, A. M., & Saxena, R. K. (2011). Mittag-Leffler functions and their applications. *Journal of Applied Mathematics, 2011*, 298628.

[41] He, J.-H. (1999). Variational iteration method–a kind of non-linear analytical technique: Some examples. *International Journal of Non-Linear Mechanics, 34*(4), 699–708.

[42] Hilfer, R. (Editor). (2000). Applications of Fractional Calculus in Physics. *World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong.*

[43] Hilfer, R. (2000). Fractional time evolution. Applications of Fractional Calculus in Physics. *World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong*, 87–130.

[44] Hilfer, R. (2002). Experimental evidence for fractional time evolution in glass forming materials. *Chemical Physics, 284*(1-2), 399–408.

[45] Hilfer, R. (2008). Threefold introduction to fractional derivatives. In: *Anomalous transport: Foundations and applications* (Klages, R., Radons, G., & Sokolov, I. M. (Editors), Wiley-VCH Verlag, Weinheim, 17–73.

[46] Hilfer, R., & Anton, L. (1995). Fractional master equations and fractal time random walks. *Physical Review E, 51*(2), R848.

[47] Hilfer, R., Luchko, Y., & Tomovski, Ž. (2009). Operational method for the solution of fractional differential equations with generalized Riemann-Liouville fractional derivatives. *Fractional Calculus and Applied Analysis, 12*(3), 299318.

[48] Hilfer, R., & Seybold, H. J. (2006). Computation of the generalized Mittag-Leffler function and its inverse in the complex plane. *Integral Transforms and Special Functions, 17*(9), 637–652.

[49] Ho, M.-W., James, L. F., & Lau, J. W. (2021). Gibbs partitions, Riemann-Liouville fractional operators, Mittag-Leffler functions, and fragmentations derived from stable subordinators. *Journal of Applied Probability, 58*(2), 314–334.

[50] Izadi, M., & Srivastava, H. M. (2021). A novel matrix technique for multi-order pantograph differential equations of fractional order. *Proceedings of the Royal Society of London, Series A, Mathematical, Physical and Engineering Sciences, 477*, 2021031.

[51] Kilbas, A. A., & Saigo, M. (2004). *H*-Transforms: Theory and Applications, *Analytical Methods and Special Functions: An International Series of Monographs in Mathematics, Vol. 9, Chapman and Hall (A CRC Press Company), Boca Raton, London and New York.*

[52] Kilbas, A. A., Saigo, M., & Saxena, R. K. (2004). Generalized Mittag-Leffler function and generalized fractional calculus operators. *Integral Transforms and Special Functions, 15*(1), 31–49.

[53] Kilbas, A. A., Saigo, M., & Saxena, R. K. (2002). Solution of Volterra integro-differential equations with generalized Mittag-Leffler function in the kernels. *Journal of Integral Equations and Applications, 14*, 377–396.

[54] Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). Theory and Applications of Fractional Differential Equations, *North-Holland Mathematical Studies, Vol. 204, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York.*

[55] Kiryakova, V. S. (1993). Generalized Fractional Calculus and Applications, *Pitman Research Notes in Mathematics, Vol. 301, Longman Scientific and Technical, Harlow (Essex).*

[56] Klafter, J., Lim, S.-C., Metzler, R. (2011). Fractional Calculus: Recent Advances, *World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong.*

[57] Koca, I. (2018). Modelling the spread of Ebola virus with Atangana-Baleanu fractional operators. *The European Physical Journal Plus, 133*(3), 1–11.

[58] Kumar, D., Singh, J., & Baleanu, D. (2017). A new analysis for fractional model of regularized long-wave equation arising in ion acoustic plasma waves. *Mathematical Methods in the Applied Sciences, 40*(15), 5642–5653.

[59] Kumar, D., Choi, J., & Srivastava, H. M. (2018). Solution of a general family of fractional kinetic equations associated with the generalized Mittag-Leffler function. *Nonlinear Functional Analysis and Applications, 23*(3), 455–471.

[60] Kumar, S., Pandey, R. K., Srivastava, H. M., & Singh, G. N. (2021). A convergent collocation approach for generalized fractional integro-differential equations using Jacobi poly-fractonomials. *Mathematics, 9*(9), 979.

[61] Li, C., Wu, Y., & Ye, R. (Eds.). (2013). Recent Advances in Applied Nonlinear Dynamics with Numerical Analysis: Fractional Dynamics, Network Dynamics, Classical Dynamics and Fractal Dynamics with Their Numerical Simulations. *Series on Interdisciplinary Mathematical Sciences, Vol. 15, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong.*

[62] Li, Z., Liu, Z., & Khan, M. A. (2020). Fractional investigation of bank data with fractal-fractional Caputo derivative. *Chaos, Solitons & Fractals, 131*, 109528.

[63] Liao, S. (2004). On the homotopy analysis method for nonlinear problems. *Applied Mathematics and Computation, 147*(2), 499–513.

[64] Liouville, J. (1832). Mémoire sur quelques quéstions de géometrie et de mécanique, et sur un nouveau genre de calcul pour résoudre ces quéstions. *J. École Polytech, 13*(21), 1–69.

[65] Lu, Q., Zhu, Y., & Lu, Z. (2019). Uncertain fractional forward difference equations for Riemann–Liouville type. *Advances in Difference Equations, 2019*(1), 1–11.

[66] Mainardi, F. (2010). Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models. *World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong.*

[67] Mainardi, F. (2020). Why the Mittag-Leffler function can be considered the Queen function of the Fractional Calculus?. *Entropy, 22*(12), 1359.

[68] Mainardi, F., & Gorenflo, R. (2008). Time-fractional derivatives in relaxation processes: a tutorial survey. *Fractional Calculus and Applied Analysis, 10*, 269–308.

[69] McBride, A. C. (1979). Fractional Calculus and Integral Transforms of Generalized Functions (No. 31). *Pitman Publishing, London.*

[70] Miller, K. S., & Ross, B. (1993). An Introduction to the Fractional Calculus and Fractional Differential Equations. *John Wiley and Sons, New York, Chichester, Brisbane, Toronto and Singapore.*

[71] Mittag-Leffler, G. M. (1903). Sur la nouvelle fonction $E_\alpha(x)$, *Comptes Rendus de l'Académie des Sciences de Paris, 137*, 554–558.

[72] Mohammed, P. O. (2019). A generalized uncertain fractional forward difference equations of Riemann–Liouville type. *Journal Mathematics Research, 11*(4), 43–50.

[73] Mohammed, P. O., Abdeljawad, T., Jarad, F., & Chu, Y.-M. (2020). Existence and uniqueness of uncertain fractional backward difference equations of Riemann–Liouville type. *Mathematical Problems in Engineering, 2020*, 6598682.

[74] Mokhtary, P., Ghoreishi, F., & Srivastava, H. M. (2016). The Müntz-Legendre Tau method for fractional differential equations. *Applied Mathematical Modelling, 40*(2), 671–684.

[75] Momani, S., Kumar, R., Srivastava, H. M., Kumar, S., & Hadid, S. (2021). A chaos study of fractional SIR epidemic model of childhood diseases. *Results in Physics, 27*, 104422.

[76] Ndairou, F., Area, I., Nieto, J. J., & Torres, D. F. (2020). Mathematical modeling of COVID-19 transmission dynamics with a case study of Wuhan. *Chaos, Solitons & Fractals, 135*, 109846.

[77] Oldham, K. B., & Spanier, J. (1974). The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order. *Academic Press, New York and London.*

[78] Olver, F. W., Lozier, D. W., Boisvert, R. F., & Clark, C. W. (Editors). (2010). NIST Handbook of Mathematical Functions (Hardback and CD-ROM). *Cambridge University Press, Cambridge, London and New York.* (see also [1]).

[79] Ortigueira, M. D. (2011). Fractional Calculus for Scientists and Engineers (Vol. 84). *Springer, Berlin, Heidelberg and New York.*

[80] Petráš, I. (2011). Fractional-Order Nonlinear Systems: Modeling, Analysis and Simulation. *Springer, Berlin, Heidelberg and New York.*

[81] Pinto, C. M. A., Carvalho, A. R. M., Baleanu, D., & Srivastava, H. M. (2019). Efficacy of the post-exposure prophylaxis and of the HIV latent reservoir in HIV infection. *Mathematics, 7*(6), 515.

[82] Podlubny, I. (1998). Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications (Vol. 198). *Academic Press, New York, London, Sydney, Tokyo and Toronto.*

[83] Postnikov, E. B. (2020). Estimation of COVID-19 dynamics "on a back-of-envelope": Does the simplest SIR model provide quantitative parameters and predictions?. *Chaos, Solitons & Fractals, 135*, 109841.

[84] Prabhakar, T. R. (1971). A singular integral equation with a generalized Mittag Leffler function in the kernel. *Yokohama Mathematical Journal, 19*, 7–15.

[85] Rachah, A., & Torres, D. F. (2017). Predicting and controlling the Ebola infection. *Mathematical Methods in the Applied Sciences, 40*(17), 6155–6164.

[86] Raina, R. K. (2005). On generalized Wright's hypergeometric functions and fractional calculus operators. *East Asian Mathematical Journal, 21*(2), 191–203.

[87] Rubin, B. (1996). Fractional Integrals and Potentials, *Pitman Monographs and Surveys in Pure and Applied Mathematics, Longman Scientific and Technical, Harlow (Essex).*

[88] Saad, K. M. (2020). New fractional derivative with non-singular kernel for deriving Legendre spectral collocation method. *Alexandria Engineering Journal, 59*(4), 1909–1917.

[89] Saad, K. M., Al-Shareef, E. H., Alomari, A. K., Baleanu, D., & Gómez-Aguilar, J. F. (2020). On exact solutions for time-fractional Korteweg-de Vries and Korteweg-de Vries-Burger's equations using homotopy analysis transform method. *Chinese Journal of Physics, 63*, 149–162.

[90] Saad, K. M., & Al-Shomrani, A. A. (2016). An application of homotopy analysis transform method for Riccati differential equation of fractional order. *Journal of Fractional Calculus and Applications, 7*(1), 61–72.

[91] Saad, K. M., Gómez-Aguilar, J. F., & Almadiy, A. A. (2020). A fractional numerical study on a chronic hepatitis C virus infection model with immune response. *Chaos, Solitons & Fractals, 139*, 110062.

[92] Salim, T. O., & Faraj, A. W. (2012). A generalization of Mittag-Leffler function and integral operator associated with fractional calculus. *Journal of Fractional Calculus and Applications, 3*(5), 1–13.

[93] Samko, S. G., Kilbas, A. A., Marichev, O. I. (1993). Fractional Integrals and Derivatives: Theory and Applications, *Gordon and Breach Science Publishers, Yverdon (Switzerland).*

[94] Sandev, T., & Tomovski, Ž. (2010). The general time fractional wave equation for a vibrating string. *Journal of Physics*

A: Mathematical and Theoretical, 43(5), 055204.

[95] Saxena, R. K., & Kalla, S. L. (2005). Solution of Volterra-type integro-differential equations with a generalized Lauricella confluent hypergeometric function in the kernels. *International Journal of Mathematics and Mathematical Sciences, 2005*(8), 1155–1170.

[96] Seybold, H. J., & Hilfer, R. (2005). Numerical results for the generalized Mittag-Leffler function. *Fractional Calculus and Applied Analysis, 8*(2), 127–139.

[97] Sharma, M., & Jain, R. (2009). A note on a generalized M-series as a special function of fractional calculus. *Fractional Calculus and Applied Analysis, 12*(4), 449–452.

[98] Shishkina, E., & Sitnik, S. (2020). Transmutations, Singular and Fractional Differential Equations with Applications to Mathematical Physics. *Mathematics in Science and Engineering, Academic Press (Elsevier Science Publishers), New York, London and Toronto.*

[99] Singh, H., Srivastava, H. M., Hammouch, Z., & Nisar, K. S. (2021). Numerical simulation and stability analysis for the fractional-order dynamics of COVID-19. Results in Physics, 20, 103722.

[100] Srivastava, H. M. (1968). On an extension of the Mittag-Leffler function. *Yokohama Mathematical Journal, 16*, 77–88.

[101] Srivastava, H. M. (2011). Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials. *Applied Mathematics & Information Sciences, 5*(3), 390–444.

[102] Srivastava, H. M. (2014). A new family of the λ-generalized Hurwitz-Lerch Zeta functions with applications. *Applied Mathematics & Information Sciences, 8*(4), 1485–1500.

[103] Srivastava, H. M. (2016). Some families of Mittag-Leffler type functions and associated operators of fractional calculus (Survey). *Turkish World Mathematical Society Journal of Pure and Applied and Mathematics, 7*, 123–145.

[104] Srivastava, H. M. (2017). Remarks on some fractional-order differential equations. *Integral Transforms and Special Functions, 28*, 560–564.

[105] Srivastava, H. M. (2019). The Zeta and related functions: Recent developments. *Journal of Advanced Engineering and Computation, 3*(1), 329–354.

[106] Srivastava, H. M. (2019). Some general families of the Hurwitz-Lerch Zeta functions and their applications: Recent developments and directions for further researches. *Proceedings of the Institute of Mathematics and Mechanics of the National Academy of Sciences of Azerbaijan, 45*, 234–269.

[107] Srivastava, H. M. (2020). Fractional-order derivatives and integrals: Introductory overview and recent developments. *Kyungpook Mathematical Journal, 60*(1), 73–116.

[108] Srivastava, H. M. (2020). Diabetes and its resulting complications: Mathematical modeling via fractional calculus. *Public Health Open Access, 4*(3), 1–5.

[109] Srivastava, H. M. (2020). Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis. *Iranian Journal of Science and Technology, Transactions A: Science, 44*(1), 327–344.

[110] Srivastava, H. M. (2021). Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations. *Journal of Nonlinear and Convex Analysis, 22*(8), 1501–1520.

[111] Srivastava, H. M., Ahmad, H., Ahmad, I., Thounthong, P., & Khan, M. N. (2021). Numerical simulation of 3-D fractional-order convection-diffusion PDE by a local meshless method. *Thermal Science, 25*(1A), 347–358.

[112] Srivastava, H. M., Alomari, A.-K. N., Saad, K. M., & Hamanah, W. M. (2021). Some dynamical models involving fractional-order derivatives with the Mittag-Leffler type kernels and their applications based upon the Legendre spectral collocation method. *Fractal and Fractional, 5*(3), 131.

[113] Srivastava, H. M., Bansal, M. K., & Harjule, P. (2018). A study of fractional integral operators involving a certain generalized multi-index Mittag-Leffler function. *Mathematical Methods in the Applied Sciences, 41*(16), 6108–6121.

[114] Srivastava, H. M., Bansal, M. K., Harjule, P. (2020). A class of fractional integral operators involving a certain general multi-index Mittag-Leffler function, *Ukrainian Mathematical Journal,* (In Press).

[115] Srivastava, H. M., Bedre, S. V., Khairnar, S. M., & Desale, B. S. (2014). Krasnosel'skii type hybrid fixed point theorems and their applications to fractional integral equations. *Abstract and Applied Analysis, 2014*, 710746; see also Corrigendum, *Abstract and Applied analysis, 2015* (2015), 467569.

[116] Srivastava, H. M., & Buschman, R. G. (1977). Convolution Integral Equations with Special Function Kernels. *Halsted Press, John Wiley and Sons, New York.*

[117] Srivastava, H. M., & Buschman, R. G. (1992). Theory and Applications of Convolution Integral Equations, *Kluwer Series on Mathematics and Its Applications, Vol. 79, Kluwer Academic Publishers, Dordrecht, Boston and London.*

[118] Srivastava, H. M.,& Choi, J. (2001). Series Associated with the Zeta and Related Functions, Vol. 530, *Kluwer Academic Publishers, Dordrecht, Boston and London.*

[119] Srivastava, H. M., & Choi, J. (2012). Zeta and q-Zeta Functions and Associated Series and Integrals. *Elsevier Science Publishers, Amsterdam, London and New York.*

[120] Srivastava, H. M., Deniz, S., & Saad, K. M. (2021). An efficient semi-analytical method for solving the generalized regularized long wave equations with a new fractional derivative operator. *Journal of King Saud University-Science, 33*(2), 101345.

[121] Srivastava, H. M., Dubey, R. S., & Jain, M. (2019). A study of the fractional-order mathematical model of diabetes and its resulting complications. *Mathematical Methods in the Applied Sciences, 42*(13), 4570–4583.

[122] Srivastava, H. M., Fernandez, A., & Baleanu, D. (2019). Some new fractional-calculus connections between Mittag–Leffler functions. *Mathematics, 7*(6), 485.

[123] Ghanbari, B., Günerhan, H., & Srivastava, H. M. (2020). An application of the Atangana-Baleanu fractional derivative in mathematical biology: A three-species predator-prey model. *Chaos, Solitons & Fractals, 138*, 109910.

[124] Srivastava, H. M., & Günerhan, H. (2019). Analytical and approximate solutions of fractional-order susceptible-infected-recovered epidemic model of childhood disease. *Mathematical Methods in the Applied Sciences, 42*(3), 935–941.

[125] Srivastava, H. M., Gupta, K. C., & Goyal, S. P. (1982). The H-Functions of One and Two Variables with Applications. *South Asian Publishers, New Delhi and Madras.*

[126] Srivastava, H. M., Harjule, P., & Jain, R. (2015). A general fractional differential equation associated with an integral operator with the H-function in the kernel. *Russian Journal of Mathematical Physics, 22*(1), 112–126.

[127] Srivastava, H. M., Jan, R., Jan, A., Deebani, W., & Shutaywi, M. (2021). Fractional-calculus analysis of the transmission dynamics of the dengue infection. *Chaos: An Interdisciplinary Journal of Nonlinear Science, 31*(5), 053130.

[128] Srivastava, H. M., & Karlsson, P. W. (1985). Multiple Gaussian Hypergeometric Series. *Halsted Press (Ellis Horwood*

Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto.

[129] Srivastava, H. M., & Kashyap, B. R. K. (1982). Special Functions in Queuing Theory and Related Stochastic Processes. *Academic Press, New York, London and Toronto.*

[130] Srivastava, H. M., Lin, S.-D., & Wang, P.-Y. (2006). Some fractional-calculus results for the $\overline{H}$-function associated with a class of Feynman integrals. *Russian Journal of Mathematical Physics, 13*(1), 94–100.

[131] Srivastava, H. M., Mandal, H., & Bira, B. (2021). Lie symmetry and exact solution of the time-fractional Hirota-Satsuma Korteweg-de Vries system. *Russian Journal of Mathematical Physics, 28*(3), 284–292.

[132] Srivastava, H. M., Manocha, H. L. (1984). A Treatise on Generating Functions. *Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto.*

[133] Srivastava, H. M., & Mohammed, P. O. (2020). A correlation between solutions of uncertain fractional forward difference equations and their paths. *Frontiers in Physics, 8.*

[134] Srivastava, H. M., Mohammed, P. O., Ryoo, C. S., & Hamed, Y. S. (2021). Existence and uniqueness of a class of uncertain Liouville-Caputo fractional difference equations. *Journal of King Saud University-Science, 33*(6), 101497.

[135] Srivastava, H. M., & Owa, S. (Editors). (1989). Univalent Functions, Fractional Calculus, and Their Applications. *Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto.*

[136] Srivastava, H. M., & Owa, S. (1992). Current Topics in Analytic Function Theory. *World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong.*

[137] Srivastava, H. M., & Saad, K. M. (2018). Some new models of the time-fractional gas dynamics equation. *Advanced Mathematical Models & Applications, 3*(1), 5–17.

[138] Srivastava, H. M., & Saad, K. M. (2020). New approximate solution of the time-fractional Nagumo equation involving fractional integrals without singular kernel. *Applied Mathematics & Information Sciences, 14*(1), 1–8.

[139] Srivastava, H. M., & Saad, K. M. (2020). Numerical simulation of the fractal-fractional Ebola virus. *Fractal and Fractional, 4*(4), 49.

[140] Srivastava, H. M., Saad, K. M., Gómez-Aguilar, J. F., & Almadiy, A. A. (2020). Some new mathematical models of the fractional-order system of human immune against IAV infection. *Mathematical Biosciences and Engineering, 17*(5), 4942–4969.

[141] Srivastava, H. M., Saad, K. M., & Khader, M. M. (2020). An efficient spectral collocation method for the dynamic simulation of the fractional epidemiological model of the Ebola virus. *Chaos, Solitons & Fractals, 140*, 110174.

[142] Srivastava, H. M., & Saxena, R. K. (2006). Operators of fractional integration and applications. *Appl. Math. Comput, 118*, 147–156.

[143] Srivastava, H. M., & Saxena, R. K. (2005). Some Volterra-type fractional integro-differential equations with a multivariable confluent hypergeometric function as their kernel. *The Journal of Integral Equations and Applications*, 199–217.

[144] Srivastava, H. M., Saxena, R. K., Pogány, T. K., & Saxena, R. (2011). Integral and computational representations of the extended Hurwitz–Lerch zeta function. *Integral Transforms and Special Functions, 22*(7), 487–506.

[145] Srivastava, H. M., Shah, F. A., & Irfan, M. (2020). Generalized wavelet quasilinearization method for solving population

growth model of fractional order. *Mathematical Methods in the Applied Sciences, 43*(15), 8753–8762.

[146] Srivastava, H. M., Shah, F. A., & Lone, W. Z. (2021). Fractional nonuniform multiresolution analysis in $L^2(\mathbb{R})$. *Mathematical Methods in the Applied Sciences, 44*, 9351–9372.

[147] Srivastava, H. M., & Tomovski, Ž. (2009). Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel. *Applied Mathematics and Computation, 211*(1), 198–210.

[148] Tomovski, Ž., Hilfer, R., & Srivastava, H. M. (2010). Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions. *Integral Transforms and Special Functions, 21*(11), 797–814.

[149] Wang, W., & Khan, M. A. (2020). Analysis and numerical simulation of fractional model of bank data with fractal–fractional Atangana–Baleanu derivative. *Journal of Computational and Applied Mathematics, 369*, 112646.

[150] Wiman, A. (1905). Über den Fundamentalsatz in der Theorie der Funcktionen $E_\alpha(x)$. *Acta Mathematica, 29*, 191–201.

[151] Wiman, A. (1905). Über die Nullstellen der Funktionen $E_\alpha(x)$. *Acta Mathematica, 29*, 217–234.

[152] Wright, E. M. (1940). The asymptotic expansion of integral functions defined by Taylor series. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences, 238*(795), 423–451.

[153] Wright, E. M. (1941). The asymptotic expansion of integral functions defined by Taylor series. II. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences, 239*, 217–232.

[154] Wright, E. M. (1948). The asymptotic expansion of integral functions and of the coefficients in their Taylor series. *Transactions of the American Mathematical Society, 64*(3), 409–438.

[155] Yang, X.-J., Baleanu, D., & Srivastava, H. M. (2015). Local Fractional Integral Transforms and Their Applications. *Academic Press (Elsevier Science Publishers), Amsterdam, Heidelberg, London and New York.*

[156] You, Z., Fečkan, M., & Wang, J. (2020). Relative controllability of fractional delay differential equations via delayed perturbation of Mittag-Leffler functions. *Journal of Computational and Applied Mathematics, 378*, 112939.

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