On Some Important Sequences and Their Relation with Pythagorean - Fermat Triples

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Abstract. *This article introduces some important sequences and their connection with the hypotenuse of the Fermat family triangle with the NSW, Jha, and Pell sequences. Defining an exciting fact about the sequence,* $\{3, 17, 99, 577, \ldots.\}$*, which consists of the first leg, has a corresponding second leg is an even perfect square such that the triple will be Pythagorean triple. Moreover, it shows the relation between Jha and NSW sequences.*

Keywords
Pythagorean Triplets, Primitive Pythagorean Triplets, Fermat family triangle, Pell sequence, Jha sequence, NSW (Newman Shank William) sequence.

1. Introduction

The origin of this Pythagorean theorem has extensive applications. For convenience, it is always assumed that $0 < a < b < h$. A triangle whose sides form a Pythagorean triple is called a Pythagorean Triangle; it is clearly a right triangle. Pythagorean triangles tell us which pairs of points with whole-number coordinates in the horizontal and vertical directions are also a whole-number distance apart. Thus, there is a one-one correspondence between Pythagorean triples and the Pythagorean Triangle.

For some natural numbers a, b, and h. A triple (a, b, h) is said to be a Pythagorean triple if it satisfies the condition:

$$a^2 + b^2 = h^2.$$

where h is the hypotenuse.

We say a triple (a, b, h) satisfying the following conditions is known as a primitive triple or primitive Pythagorean triple [1]:

$$\begin{aligned} &(i) \ \ a, b \ \ and \ \ h \in N. \\ &(ii) \ \ (a, b) = 1. \\ &(iii) \ \ a < b < h. \\ &(iv) \ \ a^2 + b^2 = h^2. \end{aligned} \tag{1}$$

In other words, a triple (a, b, h) is said to be a Primitive triple if a, b, h has no common divisor other than one. Thus, each primitive Pythagorean triple has a unique representation (a, b, h). it is clearly obvious that all primitive Pythagorean triangles can lead to infinitely many non-primitive triangles. For, if (a, b, h) is a primitive Pythagorean triple if (ka, kb, kh) is a non-primitive Pythagorean triple, where $k > 1$ is an integer. conversely, every non-primitive Pythagorean triple gives rise to primitive Pythagorean triples. Therefore, it suffices to study only primitive Pythagorean, but we focus on the particular case of the Fermat Family.

Many researchers have worked on a Pythagorean triple and its conjectures. But we notice from the different articles, that different forms of Pythagorean triples have been implemented by many researchers. Spezeski et.al. [2] have been rethinking the Pythagorean triples and developed fundaments consistent with Pythagorean triples. In [3, 4], Trivedi et. al presented an interesting representation of Pythagorean triples and mentioned some new sequences, which help to determine multiple primitive triples. Bhanotar et.al.have studied Pythagorean Triplets- Associated dual triplets and algebraic Properties, Fermat, Nepoleon Point, and its extension; Alternative, a new interesting algebraic approach using angle properties, and extended the field $Q^+ \cup \{0\}$, to explore new invention is called Dual of given triplets and some important inter-connectivity with Pythagorean triplets; Many interesting Equivalent statements and conjectures associated with Pythagorean triples have been discovered. See [5–7]. Later on, Bhanotar [8, 9] has given an extension of Pythagorean triples and associated them with other branches of mathematics, and found the relationship between Fibonacci and Lucas numbers. In [10], Jenkyns et.al. determine the least positive integer that is a part of at least n such primitive triples and obtain several conditions that help in characterizing the analogous, and Tripathi et.al [11] discussed Pythagorean triples containing a fixed integer, which leads to analyzed possible triple. See [12, 13], authors have given Fibonacci number triples and datasets on the statistical and algebraic properties of primitive Pythagorean triples. In [14, 15], Shannon et.al showed generalized golden ratios and associated pell sequences, and Trivedi et.al discussed the sum of finite terms of pell sequence in terms of NSW sequence and the interrelationship of some sequences. In a recent study, Agrawal [16] has shown interest in regards to Pythagorean Triplets before and after Pythagoras, and Bhanotar et. al. [17] mentioned an interesting fact about multiple primitive Pythagorean triplets, which help to get only multiples triples. We find some excellent interconnectivity with some necessary sequences like the Fermat sequence [6], NSW sequence [16], Jha Sequence [4], Pell Sequence [14], and some results.

This article is divided into different sections. In section 2, we give some important sequences and their interconnectivity with the Fermat family triangle. In contrast, in section 3, we show the relation of Jha [4] and NSW sequence [16] with the special case of the Pythagorean triple in which the second leg is a perfect square, and in the last section 4 conclusions are given.

2. Some important sequences and their interconnectivity with Fermat Family triplets

First, we define the Fermat family triple, for positive integers a, b and h are forms of Fermat family triplets if it satisfies the conditions $a^2 + b^2 = h^2$, and $b = a + 1$. e.g. $(20, 21, 29)$ is Fermat family triple as $a = 20, b = 20 + 1 = 21$ and $h = 29$ satisfy the conditions. More Fermat Family triplets can be seen in Tab. 1, which will help to understand the pattern with the Jha sequence as mentioned in Tab. 2.

Tab. 1: Few Triplets of Fermat Family

n	a_n	b_n	h_n
1	0	1	1
2	3	4	5
3	20	21	29
4	119	120	169
5	696	697	985
6	4059	4060	5741
7	23660	23661	33461

There are numerous Fermat family triplets, which can be found and can be analyzed. We introduce different sequences like the Pell sequence, NSW sequence, Jha sequence, and their recurrence relations.

2.1. Pell Sequence [15]

Pell numbers arise historically and most notably in the rational approximation to the square root of 2. If two integers X and Y form a solution to Pell's equation, $X^2 - 2Y^2 = \pm 1$. Then the ratio $\frac{X}{Y}$ provides a close approximation of $\sqrt{2}$. The Sequence of approximation $\left\{\frac{X}{Y} \mid \frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \ldots\ldots\right\}$. The denominator of each fraction is a Pell Number and the numerator is the sum of a Pell number and its predecessor in sequence. i.e.; the solution is of the form, $\frac{P_{n-1}+P_n}{P_n} = \sqrt{2} \approx \frac{577}{408}$. The general term of Pell's sequence is defined as follows:

$$P\left(n\right) = \frac{\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n}{2\sqrt{2}}, \; n \in N. \quad (2)$$

For $n \in N$, the pell sequence can be written as set {0, 1, 2, 5, 12, 29, 70,169....}.

2.2. Jha Sequence [15]

Jha Sequence is an infinite sequence of integers. It originates in some geometric properties of the right triangles of the Fermat family. The general term for the Jha sequence is given by

$$J(p) = \frac{\sqrt{2}}{8}\left[(3+2\sqrt{2})^{p-1} - (3-2\sqrt{2})^{p-1}\right]. \quad (3)$$

For $p \in N$, the initial terms are $0, 1, 6, 35, 204, \ldots\ldots.$

It can be observed from the Tab. 2 that the Jha sequence is the cumulative frequency of the hypotenuse of the Fermat family triplets. In general, mathematically, we can write as, for $p \in N$,

$$J\left(p\right) = \sum_{i=1}^{p} h_i. \quad (4)$$

2.3. NSW [Newman-Shank-William] Sequence [15]

It is a sequence that is closely related to the Fermat family triplet. The general term of the NSW sequence may be derived using the recurrence relation of either the Pell sequence or the Jha sequence. Some terms of NSW sequence are 1, 7, 41, 239, 1393, ...

A close look at the pattern of the terms of the Jha sequence reveals the very fact that the terms of the NSW sequence are a pairwise sum of the Jha sequence. This observation brings the logic close to writing down the general term of the NSW sequence,

$$NSW(p) = J(p+1) + J(p).$$

for all $p \in N$.

The general term of the NSW sequence is denoted as $NSW(p)$ and given by,

$$NSW(p) = \frac{1}{2}\left[\left(\sqrt{2}+1\right)\left(3+2\sqrt{2}\right)^{p-1}\right] - \frac{1}{2}\left[\left(\sqrt{2}-1\right)\left(3-2\sqrt{2}\right)^{p-1}\right]. \quad (5)$$

Now we show the interconnectivity between the above-mentioned sequences in Tab. 2, which shows how the Pell sequence co-relates with the NSW sequence and the hypotenuse of Fermat family triplets. Note that, we have mentioned $n = 0$ (zero) to maintain parity in Tab. 2.

3. Results

In this section, we give some numerical results related to the 'Jha sequence' and 'NSW sequence', which are deformed from the Pythagorean triple, whose second leg is a perfect square.

Let $T = \{(a,b,h) \mid (a,b,h)$ is a primitive triplet$\}$ satisfying properties of Eq.(1). One can notice that, if a is an odd positive integer and, $gcd(a,b) = 1$ then b is an always even positive integer.

Result-1: Prove that the m^{th} term of the sequence, $\{a_m\}$ = {3,17, 99, 577, ...} is given

Tab. 2: Interrelation between some sequences with Fermat family triplets

Fermat family				$J(n)=\sum_{i=1}^{n} h(i)$	NSW(n)= $a_n + b_n$	P(n)(Pell Sequence)
n	a_n	bn	hn			
0	-	-	-	0	0	0
1	0	1	**1**	1	1	**1**
2	3	4	**5**	6	7	2
3	20	21	**29**	35	41	**5**
4	119	120	**169**	204	239	12
5	696	697	**985**	1189	1393	**29**
6	4059	4060	**5741**	6930	8119	70
7	23660	23661	**33461**	40391	47321	**169**

by

$$a_m = \frac{1}{2}\left[\left(3+2\sqrt{2}\right)^m + \left(3-2\sqrt{2}\right)^m\right].$$

where $m \in N$.

Proof: For the sequence, $\{a_m\}$ = {3,17,99,577, ...}, the recurrence relations is given by

$$a_m = 6a_{m-1} - a_{m-2}. \tag{6}$$

where $m \geq 3$.

Let the first and second terms be $a_1 = 3$, $a_2 = 17$ respectively.

To find mth term of this sequence.

Let $a_m = t^2$, $a_{m-1} = t$ and $a_{m-2} = 1$. Therefore, we have $t^2 = 6t - 1$. Solving for t, we get, $t = 3 \pm 2\sqrt{2}$

Let us take, $t_1 = 3 + 2\sqrt{2}$ and $t_2 = 3 - 2\sqrt{2}$ with $a_1 = 3$, $a_2 = 17$.

Let

$$a_m = \alpha t_1^{m-1} + \beta t_2^{m-1}. \tag{7}$$

For $m \in N$.

For $m = 1$, we have

$$a_1 = \alpha + \beta = 3. \tag{8}$$

For $m = 2$, we get

$$a_2 = \alpha t_1 + \beta t_2 = 17.$$

$$\therefore \alpha(3+2\sqrt{2}) + \beta(3-2\sqrt{2}) = 17,$$

$$\therefore 3(\alpha+\beta) + 2\sqrt{2}(\alpha-\beta) = 17,$$

$$\therefore 9 + 2\sqrt{2}(\alpha-\beta) = 17,$$

$$\therefore 2\sqrt{2}(\alpha-\beta) = 8,$$

$$\therefore (\alpha-\beta) = 2\sqrt{2}. \tag{9}$$

Solving Eqs. (8) and (9), we get

$$\alpha = \frac{(3+2\sqrt{2})}{2}.$$

and

$$\beta = \frac{(3-2\sqrt{2})}{2}.$$

Therefore, Eq.(7) reduces to

$$a_m = \frac{(3 + 2\sqrt{2})}{2}\left(3 + 2\sqrt{2}\right)^{m-1} + \frac{(3- 2\sqrt{2})}{2}\left(3 - 2\sqrt{2}\right)^{m-1}.$$

It can be written as

$$a_m = \frac{1}{2}\left[(3+2\sqrt{2})^m + (3-2\sqrt{2})^m\right]. \tag{10}$$

where $m \in N$.

Hence, result 1 is proven. ■

Result-2. Let a be in set $\{3, 17, 99, 577, \ldots.\}$, then there exist for some $\alpha \in N$, such that the triplet $(a\ ,\ b = \alpha^2,\ h)$ is a Pythagorean primitive triplet.

Proof To show the triplet $(a\ ,\ b=\alpha^2,\ h)$ is a Pythagorean primitive triplet, for which $a, b = \frac{a^2-1}{2}$, and $h = b+1$ in N, satisfying Eq.(1) and hence, b should be an (even) perfect square.

Let

$$a \in \left\{a_m | a_m = \frac{1}{2}\left[\left(3+2\sqrt{2}\right)^m + \left(3-2\sqrt{2}\right)^m\right]\right\}.$$

where $m \in N$.

Then for some, $m_1 \in N$, the first leg of Pythagorean triple be of the form,

$$a = \frac{1}{2}\left[\left(3+2\sqrt{2}\right)^{m_1} + \left(3-2\sqrt{2}\right)^{m_1}\right].$$

Therefore, the second leg, $b = \frac{a^2-1}{2}$,

$$\therefore b = \frac{\left\{\frac{1}{2}\left[\left(3+2\sqrt{2}\right)^{m_1} + \left(3-2\sqrt{2}\right)^{m_1}\right]\right\}^2 - 1}{2}, \tag{11}$$

In order to show that b is a perfect square, we can start with some $\alpha \in N$ such that $b = \alpha^2$.

Therefore, Eq.(11) reduces to,

$$\left\{\frac{1}{2}\left[\left(3+2\sqrt{2}\right)^{m_1} + \left(3-2\sqrt{2}\right)^{m_1}\right]\right\}^2 = 2\alpha^2+1,$$

which is an odd and perfect square.

Let for some $k \in N$, we have

$$2\alpha^2 + 1 = (2k+1)^2,$$

which gives,

$$\alpha^2 = 2k\ (k+1)$$

is a perfect square.
where

$$\begin{aligned} k &= \{1, 8, 49, 288,\ 1681\ldots\}, \\ &= \{1,\ 49,\ 1681\ldots\}\,U\ \{8,\ 288,\ 9800\ldots\}, \\ &= \{1,\ 49,\ 1681\ldots\}\,U\ \{8\,(1, 36,\ 1225\ldots.)\}, \\ &= \{a_n^2 \mid a_n \text{ is an NSW sequence}\}\ \ U \\ &\quad \{8\,(1, 36,\ \ 1225\ldots.)\}, \\ &= \{a_n^2 \mid a_n \text{ is an NSW sequence}\}\ U\ \{8\ J_n^2\}, \end{aligned}$$

where J_n is a Jha-Sequence (3).

This gives the pattern of k for which $2k(k+1)$ is the perfect square. Thus, for given

$$a \in \left\{a_m | a_m = \frac{1}{2}\left[\left(3+2\sqrt{2}\right)^m + \left(3-2\sqrt{2}\right)^m\right]\right\},$$

where $m \in N$. we can find the pattern of $b = \alpha^2$ in such a way that the triple (a, b, h) will be primitive Pythagorean triplet.

For example, for a given sequence of numbers $\{3, 17, 99, 577, \ldots.\}$. The term b can be obtained as follows, and the corresponding hypotenuse is $h = b+1$, and triple (a, b, h) is primitive Pythagorean triplets.

$$\begin{aligned} b = \alpha^2 &= \{4, 144, 4900, \ldots\} \\ &= 4\left\{(1)^2, (6)^2, (35)^2, ..\right\} \end{aligned} \tag{12}$$

It can be noticed from Eq.(12), that each term of the set b is of the form, $b = \{(2J_n)^2 \mid J_n$ is Jha sequence$\}$, which proved the claim. ■

4. Conclusion

In passing the concluding remarks on the treatise of this article, we'd wish to mention that some important sequences and their interconnectivity with Fermat Family triplets are discussed and that the actual terms of various known sequences play an important role in forecasting and fixing the matter of finding m^{th} term and its recurrence relation and a few numerical results, which consistent Pythagorean triple have the second leg is even perfect square, associated with 'Jha sequence' and 'NSW sequence' are explicitly demonstrated and derived.

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