A New Family of k-Quasi Morgan-Voyce Sequences

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Abstract. *In this study, we define the k-Quasi Morgan-Voyce and k-Quasi Morgan-Voyce-Lucas sequences, and some terms of these sequences are given. We introduce the closed-form formulas that give the terms of these sequences. Also, we examine the characteristic equation and properties of these sequences. Then, we find the relations between the terms of the k-Quasi Morgan-Voyce and k-Quasi Morgan-Voyce-Lucas sequences. Also, we give these sequences generating functions, summation formulas, etc. We present special relations between the positive index term and the negative index term of these sequences. In addition, we obtain the Binet formulas in two different ways. The first method is the classic one. The second method is obtained with the help of the generating functions of the sequences. Moreover, we calculate the special identities of these sequences like Cassini and Catalan. Finally, we examine the relations of the k-Quasi Morgan-Voyce sequence with the Fibonacci, Bronze Fibonacci, Pell, Balancing, Jacobsthal, Mersenne, Oresme sequences and k-Quasi Morgan-Voyce-Lucas sequence with the Lucas, Bronze Lucas, Pell-Lucas, Balancing-Lucas, Jacobsthal-Lucas, Mersenne-Lucas, Oresme-Lucas sequences, respectively. In addition, for special k values, these sequences are associated with the sequences in OEIS.*

Keywords: *Binet Formula, Catalan Identitiy, generating function, Quasi Morgan-Voyce sequence, Sequences.*

1. Introduction

The Fibonacci and Lucas sequences are famous sequences of numbers. These sequences have intrigued scientists for a long time. Fibonacci sequences have been applied in various fields such as Cryptology [1], Phylotaxis [2], Biomathematics [3], Chemistry [4], Engineering [5], etc. Many generalizations of the Fibonacci sequence have been given. The known examples of such sequences are the k-Fibonacci, Leonardo, Horadam, k-Pell, Gaussian Fibonacci, k-Jacobsthal-Lucas, Narayana, Perrin sequences, etc. (see for details in [6–11] and [12–15]).

For $n \geq 0$, The Fibonacci numbers F_n, Bronze Fibonacci BF_n , Lucas numbers L_n, and Bronze Lucas numbers BL_n defined by the recurrence relations, respectively,

$$\begin{aligned}
F_{n+2} &= F_{n+1} + F_n, \\
BF_{n+2} &= 3BF_{n+1} + BF_n, \\
L_{n+2} &= L_{n+1} + L_n, \\
BL_{n+2} &= 3BL_{n+1} + BL_n
\end{aligned}$$

with the initial conditions $F_0 = 0, F_1 = 1, BF_0 = 0, BF_1 = 1, L_0 = 2, L_1 = 1$, and $BL_0 = 2, BL_0 = 3$.

For F_n, L_n, BF_n and BL_n the Binet formulas are given by relations, respectively

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},\ L_n = \alpha^n + \beta^n,$$
$$BF_n = \frac{\lambda^n - \psi^n}{\lambda - \psi},\ BL_n = \lambda^n + \psi^n,$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, $\lambda = \frac{3+\sqrt{13}}{2}$, and $\psi = \frac{3+\sqrt{13}}{2}$ are the roots of the characteristic equation $r^2 - r - 1 = 0$ and $v^2 - 3v - 1 = 0$. Here the number α and *lambda* are the known golden ratio and Bronze ratio, respectively.

In addition, for $n \geq 0$, the Pell numbers p_n, Pell-Lucas numbers q_n, Balancing numbers B_n, and Balancing-Lucas numbers C_n defined by the recurrence relations, respectively,

$$p_{n+2} = 2p_{n+1} + p_n,\ q_{n+2} = 2q_{n+1} + q_n,$$
$$B_{n+2} = 6B_{n+1} + B_n,\ C_{n+2} = 6C_{n+1} + C_n,$$

with the initial conditions $p_0 = 0, p_1 = 1, q_0 = 2, q_1 = 2, B_0 = 0, B_1 = 1$ and $C_0 = 2, C_1 = 6$.

For p_n, q_n, B_n and C_n the Binet formulas are given by relations, respectively,

$$p_n = \frac{\varphi^n - \omega^n}{\varphi - \omega},\ q_n = \varphi^n + \omega^n,$$
$$B_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \text{and } C_n = \gamma^n + \delta^n$$

where $\varphi = 1 + \sqrt{2}$, $\omega = 1 - \sqrt{2}$, $\gamma = 3 + 2\sqrt{2}$, and $\delta = 3 + 2\sqrt{2}$ are the roots of the characteristic equation $x^2 - 2x - 1 = 0$ and $y^2 - 6y - 1 = 0$, respectively. Here the number ϕ is the known silver ratio.

Moreover, for $n \geq 0$, the Jacobsthal numbers J_n, Jacobsthal-Lucas numbers j_n , Mersenne numbers M_n , Mersenne-Lucas numbers N_n , Oresme numbers O_n , and Oresme-Lucas numbers H_n defined by the recurrence relations, respectively,

$$J_{n+2} = J_{n+1} + 2J_n, j_{n+2} = j_{n+1} + 2j_n,$$
$$M_{n+2} = 3M_{n+1} - 2M_n,$$
$$N_{n+2} = 3N_{n+1} - 2N_n, O_{n+2} = O_{n+1} - \frac{1}{4}O_n,$$
$$H_{n+2} = H_{n+1} - \frac{1}{4}H_n,$$

with the initial conditions $J_0 = 0, J_1 = 1, j_0 = 2, j_1 = 2, M_0 = 0, M_1 = 1, N_0 = 2, N_1 = 3$, and $O_0 = 0, O_1 = 1/2, H_0 = 2, H_1 = 1$.

For $I_n, j_n, M_n, N_n, O_n,$ and H_n given by relations, respectively,

$$J_n = \frac{m^n - p^n}{m - p}, j_n = m^n + p^n, M_n = \frac{r^n - s^n}{r - s},$$
$$N_n = r^n + s^n, O_n = \frac{n}{c^n}, \text{and } H_n = \frac{1}{c^{n-1}}$$

where $m = -2, p = 1, r = 2, s = 1$, and $c_1 = c_2 = 1/2$ are the roots of the characteristic equation $a^2 - a - 2 = 0, b^2 - 3b - 2 = 0$ and $c^2 - c - 1/4 = 0$, respectively.

A. M. Morgan-Voyce introduced the Morgan-Voyce sequences. In [16], Swany defined generalized Morgan-Voyce polynomials $\{b_n\}_{n\geq 0}$ and $\{B_n\}_{n\geq 0}$ by the recurrence relations, respectively,

$$b_{n+2} = xb_{n+1} + b_n$$
$$B_{n+2} = (x+1)B_{n+1} + B_n$$

with the initial conditions $b_0 = 1, b_1 = x + 1$ and $B_0 = 1, B_1 = x + 2$.

In [17], Özgül and Sahin studies were carried out on the properties of the Morgan-Voyce polynomials. In addition, Özel et al., worked on Morgan-Voyce matrices [18].

With the help of the recurrence relation of the Fibonacci sequence, k-sequences were introduced, and these sequences have an important place in number theory [19]. In [20], Falcon and Plaza introduced the k-Fibonacci sequence and obtained many properties related to this sequence. In addition, Falcon defined the k-Lucas sequences [21]. Moreover, Falcon applied the Hankel transform to the k-Fibonacci sequence and obtained the terms of Fibonacci sequences differently [22]. Furthermore, Shannon et al defined the partial recurrence Fibonacci link and found many of its properties [23].

As seen above, many generalizations of Fibonacci and Lucas sequences have been given so far. In this study, we give new generalizations inspired by the -Fibonacci sequence and Quasi Morgan-Voyce polynomials. We call these sequences the -Quasi Morgan-Voyce and -Quasi Morgan-Voyce-Lucas sequences and denote them as $\mathcal{M}_{k,n}$, and $\mathcal{L}_{k,n}$, respectively.

We separate the article into 3 parts. In chapter 2, we define the k-Quasi Morgan-Voyce and k-Quasi Morgan-Voyce-Lucas sequences, then give the characteristic equation, the Binet formulas, and some properties for these sequences. Then we examine the relationship between k-Quasi Morgan-Voyce and k-Quasi Morgan-Voyce-Lucas sequences. In addition, we are shown the relationship of k-Quasi Morgan-Voyce and k-Quasi Morgan-Voyce-Lucas sequences for Catalan identity, Melham's identity, Vajda's identity, etc.

In chapter 3, we examine the relations of the k-Quasi Morgan-Voyce sequence with the Fibonacci, Bronze Fibonacci, Pell, Balancing, Jacobsthal, Mersenne, Oresme sequences and k-Quasi Morgan-Voyce-Lucas sequence with the Lucas, Bronze Lucas Pell-Lucas, Balancing-Lucas, Jacobsthal-Lucas, Mersenne-Lucas, Oresme-Lucas sequences, respectively. In addition, for special k values, these sequences are associated with the sequences in OEIS.

2. k-Quasi Morgan-Voyce and k-Quasi Morgan-Voyce-Lucas Sequences

In this section, we define Quasi Morgan-Voyce sequence, which is a new generalization of Fibonacci sequence inspired by Quasi and Morgan-Voyce sequences. Also, Lucas generalization of this sequence is defined by using the definition. In addition, we obtain many properties of these sequences.

For $k \in \mathbb{R}^+$, and $n \in \mathbb{N}$, the k-Quasi Morgan-Voyce $\mathcal{M}_{k,n}$, and k-Quasi Morgan-Voyce-Lucas $\mathcal{L}_{k,n}$ are defined by, respectively,

$$\mathcal{M}_{k,n+2} = (k+2)\mathcal{M}_{k,n+1} - \mathcal{M}_{k,n},$$

with $\mathcal{M}_{k,0} = 0$ and $\mathcal{M}_{k,1} = 1$,

$$\mathcal{L}_{k,n+2} = (k+2)\mathcal{L}_{k,n+1} - \mathcal{L}_{k,n},$$

with $\mathcal{L}_{k,0} = 2$ and $\mathcal{L}_{k,1} = k+2$. Then, let's give some information about the equations of these sequences.

The characteristic equation of the -Quasi Morgan-Voyce and -Quasi Morgan-Voyce-Lucas sequences is

$$r^2 - (k+2)r + 1 = 0. \tag{1}$$

The roots of the characteristic equation are as follows:

$$\begin{aligned} r_1 &= \frac{k+2+\sqrt{k^2+4k}}{2} \\ r_2 &= \frac{k+2-\sqrt{k^2+4k}}{2}. \end{aligned} \tag{2}$$

The relationship between these roots is given below

$$\begin{aligned} &r_1 + r_2 = k+2, r_1 -_2 = \sqrt{k^2+4k}, \\ &r_1^2 + r_2^2 = k^2+4k+2, \ r_1 r_2 = 1. \end{aligned}$$

The $\mathcal{M}_{k,n}$ and $\mathcal{L}_{k,n}$ values for the first four n natural numbers are given in below:

$$\begin{aligned} &\mathcal{M}_{k,0} = 0, \mathcal{M}_{k,1} = 1, \mathcal{M}_{k,2} = k+2, \\ &\mathcal{M}_{k,3} = k^2+4k+3, \\ &\mathcal{M}_{k,4} = k^3+6k^2+10k+4 \end{aligned}$$

and

$$\begin{aligned} &\mathcal{L}_{k,0} = 2, \mathcal{L}_{k,1} = k+2, \mathcal{L}_{k,2} = k^2+4k+2, \\ &\mathcal{L}_{k,3} = k^3+6k^2+9k+2, \\ &\mathcal{L}_{k,4} = k^4+8k^3+20k^2+16k+2. \end{aligned}$$

Also, the terms of the k-Quasi Morgan-Voyce $\mathcal{M}_{k,n}$, and k-Quasi Morgan-Voyce-Lucas $\mathcal{L}_{k,n}$ sequences can be found with the help of the following relations. Let $n \in \mathbb{N}^+$

$$\mathcal{M}_{k,n} = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} (k+2)^{n-1-2i}(-1)^i \tag{3}$$

and

$$\mathcal{L}_{k,n} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-i} \binom{n-i}{i} (k+2)^{n-2i}(-1)^i. \tag{4}$$

In the following theorem, the Binet formulas of the k-Quasi Morgan-Voyce $\mathcal{M}_{k,n}$ and k-Quasi Morgan-Voyce-Lucas $\mathcal{L}_{k,n}$ sequences are expressed.

Theorem 2.1. Let $n \in \mathbb{N}$. We obtain

i. $M_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}$,

ii. $\mathcal{L}_{k,n} = r_1^n + r_2^n$.

Proof. The Binet form of a sequence is as follows

$$\mathcal{M}_{k,n} = a{r_1}^n + b{r_2}^n. \quad (5)$$

The scalars a and b can be obtained by substituting the initial conditions. It is obtained by solving the given system of equations. For $n=0$, $\mathcal{M}_{k,0}=0$, and $n=1$, $\mathcal{M}_{k,1}=1$. Thus, $a = \frac{1}{\sqrt{k^2+4k}}$ and $b = \frac{-1}{\sqrt{k^2+4k}}$ are obtained. From here

$$M_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}. \quad (6)$$

The proof of the other is shown similarly. Next, we examine the relationships between the roots of the characteristic equation of these sequences and these sequences.

Theorem 2.2. We have

i. $r_1^{2i} = \frac{\mathcal{M}_{k,2i}}{k+2} r_1 \sqrt{k^2+4k} + \frac{\mathcal{L}_{k,2i-1}}{k+2}$,

ii. ${r_2}^{2i} = -\frac{\mathcal{M}_{k,2i}}{k+2} r_2 \sqrt{k^2+4k} + \frac{\mathcal{L}_{k,2i-1}}{k+2}$,

iii. ${r_1}^{2i+1} = \frac{\mathcal{M}_{k,2i}}{k+2} \sqrt{k^2+4k} + r_1 \frac{\mathcal{L}_{k,2i+1}}{k+2}$,

iv. ${r_2}^{2i+1} = -\frac{\mathcal{M}_{k,2i}}{k+2} \sqrt{k^2+4k} + r_2 \frac{\mathcal{L}_{k,2i+1}}{k+2}$,

v. $\sqrt{k^2+4k}\mathcal{M}_{k,i} + \mathcal{L}_{k,i} = 2r_1^i$,

vi. $\sqrt{k^2+4k}\mathcal{M}_{k,i} - \mathcal{L}_{k,i} = -2r_2^i$.

Proof. i. If the Binet formula is used, we obtain

$$\begin{aligned}
&\frac{\mathcal{M}_{k,zi}}{k+2} r_1 \sqrt{k^2+4k} + \frac{c_{k,zi-1}}{k+2} \\
&= r_1 \sqrt{k^2+4k} \frac{{r_1}^{2i} - {r_2}^{2i}}{(r_1 - r_2)(k+2)} + \frac{{r_1}^{2i-1} + {r_2}^{2i-1}}{k+2} \\
&= \frac{{r_1}^{2i+1} - r_1 {r_2}^{2i} + {r_1}^{2i-1} + {r_2}^{2i-1}}{k+2} \\
&= \frac{{r_1}^{2i}\left(r_1 + \frac{1}{r_1}\right) + {r_2}^{2i}\left(-r_1 + \frac{1}{r_2}\right)}{k+2} = {r_1}^{2i}
\end{aligned}$$

The proofs of the others are shown similarly.

Theorem 2.3. Let $x = r_1$ or $x = r_2$. We obtain

i. $x^a = x\mathcal{M}_{k,a} - \mathcal{M}_{k,a-1}$,

ii. $x^{2a} = x^a \mathcal{L}_{k,a} - 1$,

iii. $\mathcal{M}_{k,a(b-c)} = x^{ac}\mathcal{M}_{k,ab} - x^{ab}\mathcal{M}_{k,ac}$,

iv. $x^{ad} = \frac{x^a \mathcal{M}_{k,ad}}{\mathcal{M}_{k,a}} - \frac{\mathcal{M}_{k,a(d-1)}}{\mathcal{M}_{k,a}}$,

v. $x^a = x^b \mathcal{M}_{k,a-b+1} - x^{b-1}\mathcal{M}_{k,a-b}$,

vi. $-1 + k + 2)x + x^{2\left(2^{n+1}+1\right)} = x^{2(2^n+1)}\mathcal{L}_{k,2^{n+1}}$.

Proof. i. For $x = r_1$, we have

$$\begin{aligned}
x\mathcal{M}_{k,a} - \mathcal{M}_{k,a-1} &= r_1 \left(\frac{r_1^a - r_2^a}{r_1 - r_2}\right) - \left(\frac{r_1^{a-1} - r_2^{a-1}}{r_1 - r_2}\right) \\
&= \frac{r_1^{a-1}\left(r_1^2 - 1\right) - r_2^{a-1}\left(r_1 r_2 - 1\right)}{r_1 - r_2} = r_1^a
\end{aligned}$$

For $x = r_2$, we have

$$\begin{aligned}
x\mathcal{M}_{k,a} - \mathcal{M}_{k,a-1} &= r_2 \left(\frac{r_1^a - r_2^a}{r_1 - r_2}\right) - \left(\frac{r_1^{a-1} - r_2^{a-1}}{r_1 - r_2}\right) \\
&= \frac{r_1^a\left(r_2 - \frac{1}{r_1}\right) - r_2^a(r_2 - \frac{1}{r_2})}{r_1 - r_2} = r_2^a
\end{aligned}$$

The proofs of the others are shown similarly.

In the following theorems, we find special relations between the k-Quasi Morgan-Voyce $\mathcal{M}_{k,n}$ and k-Quasi Morgan-Voyce-Lucas $\mathcal{L}_{k,n}$ sequences.

Theorem 2.4. Let $k \in \mathbb{R}^+, m, n \in \mathbb{Z}^+$ and $m > n$. The following equations are satisfied.

i. $\mathcal{L}_{k,n} = \mathcal{M}_{k,n+1} - \mathcal{M}_{k,n-1}$,

ii. $\mathcal{L}_{k,n}^2 - (k^2+4k)\mathcal{M}_{k,n}^2 = 4$,

iii. $2\mathcal{M}_{k,m+n} = \mathcal{M}_{k,m}\mathcal{L}_{k,n} + \mathcal{L}_{k,m}\mathcal{M}_{k,n}$,

iv. $\mathcal{L}_{k,m}\mathcal{L}_{k,n} = \mathcal{L}_{k,m+n} + \mathcal{L}_{k,m-n}$,

v. $\sqrt{k^2+4k}\mathcal{M}_{k,n} = \mathcal{L}_{k,n+1} + \mathcal{L}_{k,n-1}$,

vi. $\mathcal{M}_{k,2n+2}\mathcal{L}_{k,2n+1} = \mathcal{M}_{k,4n+3} + 1$,

vii. $\mathcal{L}_{k,-n} = \mathcal{L}_{k,n}$,

viii. $\mathcal{M}_{k,-n} = -\mathcal{M}_{k,n}$.

Proof. i. If the Binet formula is used, we obtain

$$\begin{aligned}
\mathcal{M}_{k,n+1} - \mathcal{M}_{k,n-1} &= \frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2} - \frac{r_1^{n-1} - r_2^{n-1}}{r_1 - r_2} \\
&= \frac{r_1^n\left(r_1 - \frac{1}{r_1}\right) + r_2^n\left(-r_2 + \frac{1}{r_2}\right)}{r_1 - r_2} = r_1^n + r_2^n = \mathcal{L}_{k,n}
\end{aligned}$$

The proofs of the others are shown similarly.

Theorem 2.5. Let $k \in \mathbb{R}^+, m, n \in \mathbb{Z}^+$ and $m > n$. We have

i. $2\mathcal{L}_{k,m-n} = \mathcal{L}_{k,n}L_{k,m} - (k^2+4k)\mathcal{M}_{k,n}\mathcal{M}_{k,m}$,

ii. $2\mathcal{M}_{k,m-n} = L_{k,n}M_{k,m} - \mathcal{L}_{k,m}\mathcal{M}_{k,n}$,

iii. $\mathcal{L}_{k,n}\mathcal{M}_{k,m} = \mathcal{M}_{k,m-n} + \mathcal{M}_{k,m+n}$,

iv. $\mathcal{L}_{k,m+n+1} = \mathcal{L}_{k,n+1}\mathcal{M}_{k,m+1} - \mathcal{L}_{k,n}\mathcal{M}_{k,m}$,

v. $\mathcal{M}_{k,m+n+1} = \mathcal{M}_{k,m+1}\mathcal{M}_{k,n+1} - \mathcal{M}_{k,m}\mathcal{M}_{k,n}$,

vi. $\mathcal{M}_{k,3n} = (k^2+4k)\mathcal{M}_{k,n}^3 + 3\mathcal{M}_{k,n}$.

Proof. iv. If the Binet formula is used, we obtain

$$\mathcal{M}_{k,m+1}\mathcal{M}_{k,n+1} - \mathcal{M}_{k,m}\mathcal{M}_{k,n} = \frac{r_1^{m+1}-r_2^{m+1}}{r_1-r_2}\frac{r_1^{n+1}-r_2^{n+1}}{r_1-r_2} - \frac{r_1^{m}-r_2^{m}}{r_1-r_2}\frac{r_1^{n}-r_2^{n}}{r_1-r_2} = \frac{r_1^{m+n+1}(r_1-r_2) - r_2^{m+n+1}(r_1-r_2)}{(r_1-r_2)^2} = \frac{r_1^{m+n+1}-r_2^{m+n+1}}{r_1-r_2} = \mathcal{M}_{k,m+n+1}$$

The proofs of the others are shown similarly.

Theorem 2.6. Let $k \in \mathbb{R}^+$, and $m, n \in \mathbb{Z}^+$. We obtain

i. $\mathcal{M}_{k,m} + \mathcal{M}_{k,m+4n} = \mathcal{M}_{k,m+2n}\mathcal{L}_{k,2n}$,

ii. $\mathcal{M}_{k,m+3n} - \mathcal{M}_{k,m+n} = \mathcal{L}_{k,m+2n}\mathcal{M}_{k,n}$,

iii. $\mathcal{M}_{k,m+n} + \mathcal{M}_{k,m+3n} = \mathcal{L}_{k,n}\mathcal{M}_{k,m+2n}$,

iv. $\mathcal{L}_{k,m+3n} - \mathcal{L}_{k,m+n} = (k^2+4k)\mathcal{M}_{k,m+2n}\mathcal{M}_{k,n}$.

Proof. ii. If the Binet formula is used, we get

$$\begin{aligned}\mathcal{L}_{k,n}\mathcal{M}_{k,m+2n} &= (r_1^n + r_2^n)\frac{r_1^{m+2n}-r_2^{m+2n}}{r_1-r_2} \\ &= \frac{r_1^{m+3n} - r_1^n r_2^{m+2n} + r_1^{m+2n}r_2^n - r_2^{m+3n}}{r_1-r_2} \\ &= \frac{r_1^{m+sn} - r_2^{m+sn} + r_1^n r_2^n(r_1^{m+n} - r_2^{m+n})}{r_1-r_2} \\ &= \frac{r_1^{m+3n}-r_2^{m+3n}}{r_1-r_2} + \frac{r_1^{m+n}-r_2^{m+n}}{r_1-r_2} \\ &= \mathcal{M}_{k,m+3n} + \mathcal{M}_{k,m+n}.\end{aligned}$$

The proofs of the others are shown similarly.

Theorem 2.7. Let $k \in \mathbb{R}, m, n \in \mathbb{Z}^+$,and $n > m$. We have

i. $$\mathcal{M}_{k,n+3}\mathcal{M}_{k,n}^2 - \mathcal{M}_{k,n+1}^3 = \frac{1}{k^2+4k}(3\mathcal{M}_{k,n+1} - 2\mathcal{M}_{k,n+3} - \mathcal{M}_{k,n-3})$$

ii. $$\mathcal{M}_{k,n+m}^2\mathcal{L}_{k,n+m}^2 - \mathcal{M}_{k,m}^2\mathcal{L}_{k,m}^2 = \frac{1}{k^2+4k}(\mathcal{L}_{k,4m+4n} + 2\mathcal{L}_{k,2n+2m} - \mathcal{L}_{k,4m} + 4),$$

iii. $$\mathcal{M}_{k,2m+1}\mathcal{M}_{k,2n+1} = \frac{1}{k^2+4k}\left(\mathcal{L}_{k,2m+2n+2} - \mathcal{L}_{k,2n-2m}\right)$$

iv. $$\mathcal{M}_{k,n}\mathcal{L}_{k,n+m} - \mathcal{M}_{k,n+m}\mathcal{M}_{k,n-m} = \mathcal{M}_{k,2n+m} - \mathcal{M}_{k,2n} - \mathcal{M}_{k,2m} - \mathcal{M}_{k,m}$$

v. $$\mathcal{M}_{k,n}\mathcal{L}_{k,n+m} - \mathcal{L}_{k,n}\mathcal{L}_{k,n-m} = \mathcal{M}_{k,2n+m} - \mathcal{L}_{k,2-m} - \mathcal{L}_{k,m} - \mathcal{M}_{k,m}.$$

Theorem 2.8. Let $k \in \mathbb{R}$ and $a, b, c \in \mathbb{Z}^+$. The following equations are satisfied.

i. $$4\mathcal{M}_{k,a+b+c} = \mathcal{L}_{k,a}\mathcal{L}_{k,b}\mathcal{M}_{k,c} + \mathcal{M}_{k,a}\mathcal{L}_{k,b}\mathcal{L}_{k,c} + \mathcal{L}_{k,a}\mathcal{M}_{k,b}\mathcal{L}_{k,c} + (k^2+4k)\mathcal{M}_{k,a}\mathcal{M}_{k,b}\mathcal{M}_{k,c},$$

ii. $$4\mathcal{L}_{k,a+b+c} = \mathcal{L}_{k,a}\mathcal{L}_{k,b}\mathcal{L}_{k,c} + (k^2+4k)\ \mathcal{L}_{k,a}\mathcal{M}_{k,b}\mathcal{M}_{k,c} + \mathcal{M}_{k,a}\mathcal{L}_{k,b}\mathcal{L}_{k,c} + \mathcal{M}_{k,a}\mathcal{M}_{k,b}\mathcal{L}_{k,c}.$$

The proofs of Theorem 2.7 and 2.8 are shown using the Binet formulas.

Theorem 2.9. Let $k \in \mathbb{R}, a, b, c \in \mathbb{Z}^+$ and $c \neq a$. We obtain

i. $$\mathcal{L}_{k,c-a}^2 = \mathcal{L}_{k,a+b}^2 + (k^2+4k)\mathcal{M}_{k,c-a}^2\ \mathcal{L}_{k,a+b}\mathcal{M}_{k,b+c} - (k^2+4k)\mathcal{M}_{k,b+c}^2,$$

ii. $$-(k^2+4k)\mathcal{M}_{k,c-a}^2 = \mathcal{L}_{k,a+b}^2 - \mathcal{L}_{k,c-a}\mathcal{L}_{k,a+b}\mathcal{L}_{k,b+c} + \mathcal{L}_{k,b+c}^2,$$

iii. $$\mathcal{M}_{k,c-a}^2 = \mathcal{M}_{k,a+b}^2 - \mathcal{L}_{k,a-c}\mathcal{M}_{k,a+b}\mathcal{M}_{k,b+c} + \mathcal{M}_{k,b+c}^2.$$

Proof. **iii.** If the Binet formulas are used, we have

$$\mathcal{M}_{k,a+b}^2-\mathcal{L}_{k,a-c}\mathcal{M}_{k,a+b}\mathcal{M}_{k,b+c}+\mathcal{M}_{k,b+c}^2$$
$$=(\frac{r_1^{a+b}-r_2^{a+b}}{r_1-r_2})^2-(r_1^{a-c}+r_2^{a-c})$$
$$\left(\frac{r_1^{a+b}-r_2^{a+b}}{r_1-r_2}\right)\left(\frac{r_1^{b+c}-r_2^{b+c}}{r_1-r_2}\right)+(\frac{r_1^{b+c}-r_2^{b+c}}{r_1-r_2})^2$$
$$=(\frac{{r_1}^{c-a}-{r_2}^{c-a}}{r_1-r_2})^2=\mathcal{M}_{k,c-a}^2$$

The proofs of the others are shown similarly.

In the following theorems, we calculate the specific identities of the k-Quasi Morgan-Voyce $\mathcal{M}_{k,n}$ and k-Quasi Morgan-Voyce-Lucas $\mathcal{L}_{k,n}$ sequences.

Theorem 2.10. (Cassini Identity) For $n\in\mathbb{N}$, we obtain

i. $\mathcal{M}_{k,n+1}\mathcal{M}_{k,n-1}-\mathcal{M}_{k,n}^2=-1$,

ii. $\mathcal{L}_{k,n+1}\mathcal{L}_{k,n-1}-\mathcal{L}_{k,n}^2=k^2+4k$.

Proof. If the Binet formula is used, we get

i.
$$\mathcal{M}_{k,n+1}\mathcal{M}_{k,n-1}-\mathcal{M}_{k,n}^2$$
$$=\frac{{r_1}^{n+1}-{r_2}^{n+1}}{r_1-r_2}\frac{{r_1}^{n-1}-{r_2}^{n-1}}{r_1-r_2}-\frac{{r_1}^{n}-{r_2}^{n}}{r_1-r_2}\frac{{r_1}^{n}-{r_2}^{n}}{r_1-r_2}$$
$$=\frac{{r_1}^{2n}-{r_1}^{n+1}{r_2}^{n-1}-{r_2}^{n+1}{r_1}^{n-1}+{r_2}^{2n}}{(r_1-r_2)^2}$$
$$-\frac{{r_1}^{2n}-2{r_1}^{n}{r_2}^{n}+{r_2}^{2n}}{(r_1-r_2)^2}=\frac{(r_1r_2)^n\frac{-r_1}{r_2}}{(r_1-r_2)^2}+\frac{(r_1r_2)^{n-r_2}}{(r_1-r_2)^2}$$
$$+\frac{2\cdot(r_1r_2)^n}{(r_1-r_2)^2}=-1.$$

ii.
$$\mathcal{L}_{k,n+1}\mathcal{L}_{k,n-1}-\mathcal{L}_{k,n}^2$$
$$=(r_1^{n+1}+r_2^{n+1})(r_1^{n-1}+r_2^{n-1})$$
$$-(r_1^n+r_2^n)(r_1^n+r_2^n)$$
$$=r_1^{2n}+r_1^{n+1}r_2^{n-1}+r_2^{n+1}r_1^{n-1}+r_2^{2n}$$
$$-{r_1}^{2n}-2{r_1}^{n}{r_2}^{n}-{r_2}^{2n}=k^2+4k$$

Theorem 2.11. (Catalan Identity) For $n,r\in\mathbb{N}$, we have

i. $\mathcal{M}_{k,n+r}\mathcal{M}_{k,n-r}-\mathcal{M}_{k,n}^2=-\mathcal{M}_{k,r}^2$,

ii. $\mathcal{L}_{k,n+r}\mathcal{L}_{k,n-r}-\mathcal{L}_{k,n}^2=(k^2+4k)\mathcal{M}_{k,r}^2$.

Theorem 2.12. (D'ocagne's Identity) For n,r natural numbers, and $r\le n$, we have

i. $\mathcal{M}_{k,n+1}\mathcal{M}_{k,r}-\mathcal{M}_{k,n}\mathcal{M}_{k,r+1}=-\mathcal{M}_{k,n-r}$,

ii. $\mathcal{L}_{k,n+1}\mathcal{L}_{k,r}-\mathcal{L}_{k,n}\mathcal{L}_{k,r+1}=(k^2+4k)\mathcal{M}_{k,n-r}$.

Theorem 2.13. (Vajda's Identity) For $n,i,j\in\mathbb{N}$, we have

i. $\mathcal{M}_{k,n+i}\mathcal{M}_{k,n+j}$-$\mathcal{M}_{k,n}\mathcal{M}_{k,n+i+j}$=$\mathcal{M}_{k,i}\mathcal{M}_{k,j}$,

ii.
$$\mathcal{L}_{k,n+i}\mathcal{L}_{k,n+j}-\mathcal{L}_{k,n}\mathcal{L}_{k,n+i+j}$$
$$=-(k^2+4k)\mathcal{M}_{k,i}\mathcal{M}_{k,j}.$$

Theorem 2.14. (Halton Identity) For $n,i,j\in\mathbb{N}$, we have

i.
$$\mathcal{M}_{k,n+i}\mathcal{M}_{k,n-i}-\mathcal{M}_{k,n+j}\mathcal{M}_{k,n-j}$$
$$=\frac{1}{k^2+4k}(\mathcal{L}_{k,2j}-\mathcal{L}_{k,2i}),$$

ii.
$$\mathcal{L}_{k,n+i}\mathcal{L}_{k,n-i}-\mathcal{L}_{k,n+j}\mathcal{L}_{k,n-j}$$
$$=2\mathcal{L}_{k,2n}+\mathcal{L}_{k,2i}+\mathcal{L}_{k,2j}.$$

Theorem 2.15. (Padilla Identity) For $n\in\mathbb{N}$, we have

i.
$$\mathcal{M}_{k,n+2}^3+\mathcal{M}_{k,n-1}^3-3\mathcal{M}_{k,n}\mathcal{M}_{k,n+1}\mathcal{M}_{k,n+2}$$
$$=\frac{1}{k^2+4k}\left(\mathcal{M}_{k,3n+6}-3\mathcal{M}_{k,3n+3}+\mathcal{M}_{k,3n-3}\right)$$
$$+3\mathcal{M}_{k,2n+3}-3\mathcal{M}_{k,n+2}+3\mathcal{M}_{k,n+1}),$$

ii.
$$\mathcal{L}_{k,n+2}^3+\mathcal{L}_{k,n-1}^3-3\mathcal{L}_{k,n}\mathcal{L}_{k,n+1}\mathcal{L}_{k,n+2}$$
$$=\mathcal{L}_{k,3n+6}+3\mathcal{L}_{k,n+2}+\mathcal{L}_{k,3n-3}+3\mathcal{L}_{k,n-1}$$
$$+(k^2+4k)(-\mathcal{M}_{k,3n+3}-3\mathcal{M}_{k,n-1}$$
$$+3\mathcal{M}_{k,n+1}+3\mathcal{M}_{k,2n+3}).$$

Theorem 2.16. (Melham's Identity) For $n\in\mathbb{N}$, we have

i.
$$\mathcal{M}_{k,n+1}\mathcal{M}_{k,n+2}\mathcal{M}_{k,n+6}-\mathcal{M}_{k,n}^3$$
$$=\frac{1}{k^2+4k}\left(\mathcal{M}_{k,3n+9}-\mathcal{M}_{k,3n}+3\mathcal{M}_{k,n}-\mathcal{M}_{k,n-3}\right.$$
$$\left.-\mathcal{M}_{k,n+5}-\mathcal{M}_{k,n+7}\right),$$

ii.
$$\mathcal{L}_{k,n+1}\mathcal{L}_{k,n+2}\mathcal{L}_{k,n+6}-\mathcal{L}_{k,n}^3=\mathcal{L}_{k,3n+9}$$
$$-3\mathcal{L}_{k,n}+\mathcal{L}_{k,n-3}+\mathcal{L}_{k,n+5}+\mathcal{L}_{k,n+7}-\mathcal{L}_{k,3n}.$$

Theorem 2.17. (Gelin-Cesaro's Identity) For $n \in \mathbb{N}$, we have

i.
$$\begin{aligned}&\mathcal{M}_{k,n+2}\mathcal{M}_{k,n+1}\mathcal{M}_{k,n-1}\mathcal{M}_{k,n-2} - \mathcal{M}_{k,n}^4\\ &= \frac{1}{(k^2+4k)^2}(-L_{k,2n+4} - L_{k,2n+2} + L_{k,6}\\ &- \mathcal{L}_{k,2n-2} - \mathcal{L}_{k,2} - \mathcal{L}_{k,2n-4} + 4\mathcal{L}_{k,2n} - 5),\end{aligned}$$

ii.
$$\begin{aligned}&\mathcal{L}_{k,n+2}\mathcal{L}_{k,n+1}\mathcal{L}_{k,n-1}\mathcal{L}_{k,n-2} - \mathcal{L}_{k,n}^4\\ &= \mathcal{L}_{k,2n+4} + \mathcal{L}_{k,2n+2} + \mathcal{L}_{k,6} + \mathcal{L}_{k,2n-2} + \mathcal{L}_{k,2}\\ &+ \mathcal{L}_{k,2n-4} - 4\mathcal{L}_{k,2n} - 5.\end{aligned}$$

The proofs of Theorem 2.11-2.17 are shown similar way to Theorem 2.10.

In the following theorems, we obtain special sum formulas of the k-Quasi Morgan-Voyce $\mathcal{M}_{k,n}$ and k-Quasi Morgan-Voyce-Lucas $\mathcal{L}_{k,n}$ sequences.

Theorem 2.20. For $n \in \mathbb{N}$, we have

i. $\sum_{s=0}^{n} \mathcal{M}_{k,s} = \frac{(k+1)\mathcal{M}_{k,n} - \mathcal{M}_{k,n-1} + 1}{k}$,

ii. $\sum_{s=0}^{n} \mathcal{L}_{k,s} = \frac{(k+1)\mathcal{L}_{k,n} - \mathcal{L}_{k,n-1} + k}{k}$.

Proof. **i.** From the definition of the k-Quasi Morgan-Voyce sequence, the following equations are written:

$$\begin{aligned}\mathcal{M}_{k,2} &= (k+2)\mathcal{M}_{k,1} - \mathcal{M}_{k,0},\\ \mathcal{M}_{k,3} &= (k+2)\mathcal{M}_{k,2} - \mathcal{M}_{k,1},\\ &\dots\\ \mathcal{M}_{k,n} &= (k+2)\mathcal{M}_{k,n-1} - \mathcal{M}_{k,n-2}.\end{aligned}$$

So, we have

$$\begin{aligned}-1 + \sum_{s=0}^{n} \mathcal{M}_{k,s} &= (k+2)\sum_{s=1}^{n-1} \mathcal{M}_{k,s}\\ &- \sum_{s=0}^{n-2} \mathcal{M}_{k,s}\\ 1 + \sum_{s=0}^{n} \mathcal{M}_{k,s} &= (-\mathcal{M}_{k,n} - \mathcal{M}_{k,0})(k+2)\\ &+ (k+2)\sum_{s=0}^{n} \mathcal{M}_{k,s}\\ &- (-M_{k,n} - M_{k,n-1} + \sum_{s=0}^{n} M_{k,s})\end{aligned}$$

Thus, we obtain

$$\sum_{s=0}^{n} M_{k,s} = \frac{(k+1)M_{k,n} - M_{k,n-1} + 1}{k}.$$

The proofs of the others are shown similarly.

Theorem 2.21. We obtain

i. $\sum_{s=0}^{n} M_{k,2s} = \frac{(k+2)M_{k,2n+1} - 2M_{k,2n} - k - 2}{k^2+4k}$,

ii.
$$\begin{aligned}&\sum_{s=0}^{n} \mathcal{M}_{k,2s+1}\\ &= \frac{(k^2+4k+2)M_{k,2n+1} - (k+2)M_{k,2n} - 2}{k^2+4k},\end{aligned}$$

iii. $\sum_{s=0}^{n} \mathcal{L}_{k,2s} = \frac{(k+2)\mathcal{L}_{k,2n+1} - 2\mathcal{L}_{k,2n} + k^2 + 4k}{k^2+4k}$,

iv. $\sum_{s=0}^{n} \mathcal{L}_{k,2s+1} = \frac{(k^2+4k+2)\mathcal{L}_{k,2n+1} - (k+2)\mathcal{L}_{k,2n}}{k^2+4k}$.

Proof. The proofs of the theorem are shown similar to Theorem 2.20.

Theorem 2.22 For $b, p, r, n \in \mathbb{N}$, and $b > r$, we obtain

i.
$$\begin{aligned}&(-1)^n \mathcal{M}_{k,bn+2n+r}\\ &= \sum_{j=0}^{n} \binom{n}{j} (-1)^j (k+2)^j \mathcal{M}_{k,bn+r+j},\end{aligned}$$

ii.
$$\begin{aligned}&(-1)^n \mathcal{L}_{k,bn+2n+r}\\ &= \sum_{j=0}^{n} \binom{n}{j} (-1)^j (k+2)^j \mathcal{L}_{k,bn+r+j},\end{aligned}$$

iii. $\sum_{j=0}^{n} \frac{\mathcal{M}_{k,bj+r}}{p^j} = \frac{1}{1-p\mathcal{L}_{k,b}+p^2} \frac{1}{p^n} (-p\mathcal{M}_{k,bn+b+r} + \mathcal{M}_{k,bn+r} - p^{n+1}\mathcal{M}_{k,b-r} + p^{n+2}\mathcal{M}_{k,r})$,

iv.
$$\begin{aligned}\sum_{j=0}^{n} \frac{\mathcal{L}_{k,bj+r}}{p^j} &= \frac{1}{1-p\mathcal{L}_{k,b}+p^2}\frac{1}{p^n}(\mathcal{L}_{k,bn+r}\\ &+ p^{n+2}\mathcal{L}_{k,r} - p\mathcal{L}_{k,bn+b+r} - p^{n+1}\mathcal{L}_{k,b-r}).\end{aligned}$$

Proof. If Binet formulas, definitions, and geometric series are used, we obtain

$$\sum_{j=0}^{n}\binom{n}{j}(-1)^j(k+2)^j\mathcal{M}_{k,bn+r+j}$$
$$=\sum_{j=0}^{n}\binom{n}{j}(-1)^j(k+2)^j\frac{r_1^{bn+r+j}-r_2^{bn+r+j}}{r_1-r_2}$$
$$=\frac{1}{r_1-r_2}[r_1^{bn+r}(1-(k+2)r_1)^n$$
$$-r_2^{bn+r}(1-(k+2)r_2)^n]$$
$$=\frac{(-1)^n}{r_1-r_2}(r_1^{bn+2n+r}-r_2^{bn+2n+r})$$
$$=(-1)^n\mathcal{M}_{k,bn+2n+r}$$

The proofs of the others are shown similarly.

In the following theorems, we give special generating functions of the k-Quasi Morgan-Voyce $\mathcal{M}_{k,n}$ and k-Quasi Morgan-Voyce-Lucas $\mathcal{L}_{k,n}$ sequences. In addition, we obtain Binet formulas of $\mathcal{M}_{k,n}$ and $\mathcal{L}_{k,n}$ sequences with the help of generating functions.

Theorem 2.23. The generating functions for k-Quasi Morgan-Voyce and k-Quasi Morgan-Voyce-Lucas sequences are given as follows, respectively,

i. $m(x)=\sum_{n=0}^{\infty}\mathcal{M}_{k,n}x^n=\frac{x}{1-(k+2)x+x^2}$,

ii. $l(x)=\sum_{n=2}^{\infty}\mathcal{L}_{k,n}x^n=\frac{-xk-2x+2}{1-(k+2)x+x^2}$.

Proof. i. The following equations are written for the k-Quasi Morgan-Voyce sequence:

$$=x+(k+2)\sum_{n=2}^{\infty}\mathcal{M}_{k,n-1}x^n$$
$$m(x)=\sum_{n=0}^{\infty}\mathcal{M}_{k,n}x^n=x+\sum_{n=2}^{\infty}\mathcal{M}_{k,n}x^n$$
$$-\sum_{n=2}^{\infty}M_{k,n-2}x^n$$
$$=x+x(k+2)\sum_{n=1}^{\infty}\mathcal{M}_{k,n}x^n$$
$$-x^2\sum_{n=0}^{\infty}\mathcal{M}_{k,n}x^n$$

Thus, we have

$$m(x)=\frac{x}{1-(k+2)x+x^2}.$$

The proofs of the others are shown similarly.

Theorem 2.24. For $\mathcal{M}_{k,n}$ and $\mathcal{L}_{k,n}$ sequences, the Binet formulas can be obtained with the help of the generating functions.

Proof. With the help of the roots of the characteristic equation of these sequences, the roots of the $1-(k+2)x+x^2=0$ equation become $1/r_1$, and $1/r_2$. For $\mathcal{M}_{k,n}$, we have

$$\frac{x}{1-(k+2)x+x^2}$$
$$=\frac{1}{r_1-r_2}\frac{1}{1-r_1x}-\frac{1}{r_1-r_2}\frac{1}{1-r_2x}$$
$$=\frac{1}{r_1-r_2}\sum_{n=0}^{\infty}{r_1}^nx^n-\frac{1}{r_1-r_2}\sum_{n=0}^{\infty}{r_2}^nx^n$$
$$=\sum_{n=0}^{\infty}(\frac{{r_2}^n-{r_2}^n}{r_1-r_2})x^n=\sum_{n=0}^{\infty}M_{k,n}x^n.$$

Similarly, the Binet formula of the sequence $\mathcal{L}_{k,n}$ is found.

Theorem 2.25. For $a,b\in\mathbb{N}$, and $b>a$, we obtain

i. $\sum_{i=0}^{\infty}\mathcal{M}_{k,bn}x^n=\frac{x\mathcal{M}_{k,b}}{1-x\mathcal{L}_{k,b}+x^2}$,

ii. $\sum_{i=0}^{\infty}\mathcal{L}_{k,bn}x^n=\frac{2-x\mathcal{L}_{k,a}}{1-x\mathcal{L}_{k,a}+x^2}$,

iii. $\sum_{i=0}^{\infty}\mathcal{M}_{k,an+b}x^n=\frac{\mathcal{M}_{k,b}-x\mathcal{M}_{k,b-a}}{1-x\mathcal{L}_{k,a}+x^2}$,

iv. $\sum_{i=0}^{\infty}\mathcal{L}_{k,an+b}x^n=\frac{\sqrt{k^2+4k}\mathcal{M}_{k,b}-x\mathcal{M}_{k,b-a}}{1-x\mathcal{L}_{k,a}+x^2}$,

v. $\sum_{n=0}^{\infty}\frac{\mathcal{M}_{k,bn}}{n!}x^n=\frac{e^{r_1^bx}-e^{r_2^bx}}{r_1-r_2}$,

vi. $\sum_{n=0}^{\infty}\frac{\mathcal{L}_{k,bn}}{n!}x^n=e^{r_1^bx}+e^{r_2^bx}$.

Proof. i. If the Binet formula is used, we get

$$\sum_{i=0}^{\infty}\mathcal{M}_{k,an}x^n=\sum_{n=0}^{\infty}\frac{{r_1}^{bn}-{r_2}^{bn}}{r_1-r_2}x^n=$$
$$\frac{1}{r_1-r_2}\sum_{n=0}^{\infty}(r_1^bx)^n-\frac{1}{r_1-r_2}\sum_{n=0}^{\infty}(r_2^bx)^n$$
$$=\frac{1}{r_1-r_2}\left(\frac{1}{1-r_1^bx}-\frac{1}{1-r_2^bx}\right)=\frac{x\mathcal{M}_{k,b}}{1-xC_{k,b}+x^2}$$

The proofs of the others are shown similarly.

3. Relations between Special Sequences

In this chapter, we examine the relations of the k-Quasi Morgan-Voyce sequence with the Fibonacci, Bronze Fibonacci, Pell, Balancing, Jacobsthal, Mersenne, Oresme sequences and k-Quasi Morgan-Voyce-Lucas sequence with the Lucas, Bronze Lucas, Pell-Lucas, Balancing-Lucas, Jacobsthal-Lucas, Mersenne-Lucas, Oresme-Lucas sequences, respectively. In addition, for special k values, these sequences are associated with the sequences in OEIS.

Theorem 3.1. For the $k=1, k=5$ values, the following relations can be written between the k-Quasi Morgan-Voyce sequence and Fibonacci sequence F_n, k-Quasi Morgan-Voyce-Lucas sequence and Lucas sequence L_n, respectively;

i. $\mathcal{M}_{1,n}=F_{2n}$ and $\mathcal{M}_{5,n}=\frac{F_{4n}}{3}$,

ii. $\mathcal{L}_{1,n}=L_{2n}$ and $\mathcal{L}_{5,n}=L_{4n}$.

Proof. i. The Binet formula of the k-Quasi Morgan-Voyce sequence is

$$\mathcal{M}_{k,n}=\frac{r_1^n-r_2^n}{r_1-r_2}=\frac{(\frac{k+2+\sqrt{k^2+4k}}{2})^n-(\frac{k+2-\sqrt{k^2+4k}}{2})^n}{\sqrt{k^2+4k}}.$$

For $k=1$ and $k=5$, the following relations can be written:

$$\mathcal{M}_{1,n}=\frac{(\frac{3+\sqrt{5}}{2})^n-(\frac{3-\sqrt{5}}{2})^n}{\sqrt{5}}=\frac{(\frac{1+\sqrt{5}}{2})^{2n}-(\frac{1-\sqrt{5}}{2})^{2n}}{\sqrt{5}}$$

and

$$\mathcal{M}_{5,n}=\frac{(\frac{7+3\sqrt{5}}{2})^n-(\frac{7-3\sqrt{5}}{2})^n}{3\sqrt{5}}=\frac{(\frac{1+\sqrt{5}}{2})^{4n}-(\frac{1-\sqrt{5}}{2})^{4n}}{3\sqrt{5}}.$$

Thus, we obtain

$$\mathcal{M}_{1,n}=F_{2n} \text{ and } \mathcal{M}_{5,n}=\frac{F_{4n}}{3}.$$

Proof. ii. The Binet formula of the k-Quasi Morgan-Voyce-Lucas sequence is

$$\begin{aligned}\mathcal{L}_{k,n}&=r_1^n+r_2^n\\&=(\frac{k+2+\sqrt{k^2+4k}}{2})^n+(\frac{k+2-\sqrt{k^2+4k}}{2})^n\end{aligned}$$

For $k=1$ and $k=5$, the following relations can be written:

$$\begin{aligned}\mathcal{L}_{1,n}&=(\frac{3+\sqrt{5}}{2})^n+(\frac{3-\sqrt{5}}{2})^n\\&=(\frac{1+\sqrt{5}}{2})^{2n}+(\frac{1-\sqrt{5}}{2})^{2n}\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_{5,n}&=(\frac{7+3\sqrt{5}}{2})^n+(\frac{7-3\sqrt{5}}{2})^n\\&=(\frac{1+\sqrt{5}}{2})^{4n}+(\frac{1-\sqrt{5}}{2})^{4n}\end{aligned}$$

Thus, we obtain

$$\mathcal{L}_{1,n}=L_{2n} \text{ and } \mathcal{L}_{5,n}=L_{4n}.$$

Theorem 3.2. For the $k=4$ value, the following relations can be written between the k-Quasi Morgan-Voyce sequence and Bronze Fibonacci sequence p_n, k-Quasi Morgan-Voyce-Lucas sequence and Bronze Lucas sequence q_n, respectively;

i. $\mathcal{M}_{4,n}=\frac{p_{2n}}{2}$,

ii. $\mathcal{L}_{4,n}=q_{2n}$.

Proof. The proofs are shown in a similar to Theorem 3.1.

Theorem 3.4. For the $k=9/4$ value, the following relations can be written between the k-Quasi Morgan-Voyce sequence and Jacobsthal sequence J_n , k-Quasi Morgan-Voyce-Lucas sequence and Jacobsthal-Lucas sequence j_n, respectively;

i. $\mathcal{M}_{\frac{9}{4},n}=\frac{1}{5}\frac{1}{4^{n-1}}J_{4n}$,

ii. $\mathcal{L}_{\frac{9}{4},n}=\frac{1}{4^n}j_{4n}$.

Proof. The proofs are shown in a similar to Theorem 3.1.

Theorem 3.5. For the $k=49/8$ value, the following relations can be written between the k-Quasi Morgan-Voyce sequence and Mersenne sequence M_n, k-Quasi Morgan-Voyce-Lucas sequence and Mersenne-Lucas sequence N_n, respectively;

i. $\mathcal{M}_{\frac{49}{8},n} = \frac{1}{8^{n-1}}\frac{1}{63}M_{6n}$,

ii. $\mathcal{L}_{\frac{49}{8},n} = \frac{1}{8^n}N_{6n}$.

Proof. The proofs are shown in a similar to Theorem 3.1.

Theorem 3.6. For the $k = 4$ value, the following relations can be written between the k-Quasi Morgan-Voyce sequence and Balancing sequence B_n , k-Quasi Morgan-Voyce-Lucas sequence and Balancing-Lucas sequence C_n, respectively;

i. $\mathcal{M}_{4,n} = B_n$,

ii. $\mathcal{L}_{4,n} = C_n$.

Proof. The proofs are shown in a similar to Theorem 3.1.

Theorem 3.7. For the $k = 2$ value, the following relations can be written between the k-Quasi Morgan-Voyce sequence and Oresme sequence O_n , k-Quasi Morgan-Voyce-Lucas sequence and Oresme-Lucas sequence H_n, respectively;

i. $\mathcal{M}_{2,n} = \frac{2}{3n}(4^n - 1)O_n$,

ii. $\mathcal{L}_{2,n} = \frac{1}{2}(4^n + 1)H_n$.

Proof. The proofs are shown in a similar to Theorem 3.1.

Theorem 3.8. The following relations are provided for some k values.

i. For $k = 3, \mathcal{M}_{3,n} = C_n$ and $\mathcal{L}_{3,n} = D_n$,

ii. For $k = 6, \mathcal{M}_{6,n} = E_n$ and $\mathcal{L}_{6,n} = F_n$,

iii. For $k = 6, \mathcal{M}_{7,n} = G_n$ and $\mathcal{L}_{7,n} = H_n$,

Here the C_n, D_n, E_n, F_n, G_n and H_n sequences are the A004254, A003501, A001090, A086903, A143325, and A056918 sequences in OEIS, respectively.

Proof. The proofs are shown in a similar to Theorem 3.1.

4. Conclusions

In this paper, we defined the k-Quasi Morgan-Voyce and k-Quasi Morgan-Voyce-Lucas sequences. Then, we found the main features of these sequences. Also, we examined the relationships between the terms of these sequences. We again associated the sum of squares of consecutive terms of these series with the sequences. In addition, unlike the known, we obtained Binet formulas with the help of the generating functions. We did exercises on the binomial sum formulas of these sequences. Moreover, we associated k-Quasi Morgan-Voyce and k-Quasi Morgan-Voyce-Lucas sequences with Fibonacci, Bronze Fibonacci, Pell, Balancing, Jacobsthal, Mersenne, Oresme and Lucas, Bronze Lucas, Pell-Lucas, Balancing-Lucas, Jacobsthal-Lucas, Mersenne-Lucas, Oresme-Lucas numbers, respectively. Furthermore, for special values, we associated these sequences with the sequence of the OEİS. If this study is examined, such features can be found in other sequences such as Horadam, and Mersenne sequences. Additionally, studies can be done on special transformations of these sequences such as the Catalan transform of the k-Quasi Morgan-Voyce and k-Quasi Morgan-Voyce-Lucas sequences and k-Quasi Morgan-Voyce and k-Quasi Morgan-Voyce-Lucas quaternions.

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